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MATHEMATICAL TEXTS FOR SCHOOLS

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Plane and Spherical Trigonometry

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Four-Place Tables of Logarithms

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PLANE GEOMETRY

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SCHOOL OF YALE UNIVERSITY

GINN AND COMPANY

BOSTON · NEW YORK · CHICAGO · LONDON

912.6

The Athenæum Press
GINN AND COMPANY · PRO-
PRIETORS · BOSTON · U.S.A.

PREFACE

The addition of another text to the long list of existing geometry texts calls for an explanation of the motives underlying its preparation.

Briefly stated, this text aims to effect a compromise between the extreme demands of certain reformers and the equally untenable position of overconservative writers. Our educational troubles cannot be cured by a complete break with the past, nor by ignoring the legitimate demands of our times. Accordingly, the authors have tried to endow the subject with life and reality, and at the same time to retain, as far as possible, that spirit of careful reasoning for which geometry has always been famous.

To accomplish this double purpose, the authors depend mainly on the following features:

1. *A Preliminary Course precedes the Demonstrative Course.* It had long been the experience of the authors that many pupils become permanently discouraged at the very beginning of demonstrative geometry by the simultaneous appearance of too many difficulties. The Preliminary Course is intended to serve as a preparation for formal geometry by vitalizing the content of all definitions by abundant illustration and discussion; by cultivating skill in the use of ruler and compasses through interesting drawing exercises; by presenting exercises requiring for their solution simple reasoning and inference; and by developing gradually the conviction that formal proof is necessary for further advance.

The actual time required by the Preliminary Course depends somewhat on the class and on the mode of procedure. A laboratory plan with individual notebooks may be followed, or the exercises may be solved on the blackboard. In either case the Preliminary Course can be completed in five or six weeks.

Careful tests in many classrooms have proved that the introductory work results in an actual saving of time. In fact, the entire book does not require more than the usual allotted time.

2. *The Demonstrative Course is built up not only in a topical but also in a psychological order.* The authors have paid great attention to a natural, progressive order of topics, and have endeavored to arrange a sequence of theorems which should be teachable. The assumptions to be used in each Book are summarized at the beginning, and each difficult topic is approached by way of an informal discussion. Consideration of the incommensurable case is made unnecessary by the sequence of theorems and by assumptions which replace the usual attempts at demonstration. Hypothetical constructions are avoided. Moreover, the whole course may be completed without a formal consideration of the theory of limits. At the end of Book V, however, teachers preferring a formal treatment of limits will find a clear presentation of this topic.

3. *The methods embodied in the text aim to make the pupil independent of the printed page.* In too many cases students merely verify and reproduce the statements of the book. A creative spirit is absolutely essential in geometry, and in this text the attempt is made to cultivate a spirit of discovery from the very beginning. At the beginning of each new topic complete or nearly complete proofs are given. As the student advances, proofs are either omitted (pp. 112, 148, 167), or are given in outline (pp. 119, 273), or are suggested by an analysis (pp. 161, 268). In this way, even after a rather extensive discussion in the classroom, work is left for the student to do.

4. *The various types of exercises receive practically equal attention.* Constructions, computations, and original theorems will be found throughout the text. Many of these carefully graded exercises can be worked at sight. Special attention is called to the use of *composite figures* (see p. 116, Ex. 8; p. 117, Ex. 9; p. 141, Ex. 11; p. 175, Ex. 14; p. 213, Ex. 11; etc.). These have the important function of enabling the student to

make his own discoveries. To aid in the discovery of geometric relations, tables of methods are given from time to time (see pp. 76, 82, 85, 93, 118).

5. *The list of applied problems is extensive but not excessive.* A proper balance between theory and practice is of the greatest importance, if the demonstrative work is not to be seriously impaired. Accordingly, the attempt has been made to give preference to such applications as have probably come within the pupil's range of experience. Problems requiring an unusual amount of explanation, even if interesting in themselves, have been excluded.

6. *The text provides a minor and a major course.* Thus the teacher may omit, without modifying the remainder of the work, any or all of the following pages: Part II of the Preliminary Course (pp. 53–68), the sections on coördinates (pp. 177–180), the trigonometric work (pp. 259–264), and the formal work on limits (pp. 309–316). A number of theorems may be treated as exercises, for example, those on pp. 136–139, 152, 161, 225, 268, 277–283.

The list of theorems is considerably reduced, but it is entirely adequate to meet the requirements of higher institutions.

The authors note with pleasure that their work seems to be in agreement with the suggestions of various associations and committees, notably with the report of the Committee on a National Geometry Syllabus.

Acknowledgments are due to many friends for valuable suggestions and criticisms. In particular, the authors wish to express their special indebtedness to their colleagues in Rochester and Newark for their interest in the preparation of this text, and for their willingness to try out the manuscript in their classes.

THE AUTHORS

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PLANE GEOMETRY

PRELIMINARY COURSE

1. Origin of Geometry. Egyptian Geometry. Geometry is one of the most ancient of all arts and sciences It arose in Babylonia and Egypt in connection with such practical activities as building, surveying, navigating, etc. In fact, the word "geometry" means *earth measurement*. Herodotus, a Greek historian who traveled in Egypt, says that the annual overflowing of the Nile changed many boundaries in the adjoining farm land. Thus it became necessary to measure the land of each taxpayer every year in order that taxes might be properly adjusted. In this way, he claims, geometry originated in Egypt, and all the classical writers agree with him in calling Egypt the home of geometry.

Much new light was thrown on early geometry by the discovery, in recent years, of Babylonian inscriptions and Egyptian papyri. As early as 1700 B.C. an Egyptian scribe, Ahmes, wrote a mathematical treatise containing a number of geometric rules.

Even without such written records the enormous architectural works of the ancients — their pyramids, obelisks, temples, palaces, and canals — would indicate a very respectable insight into geometric relations.

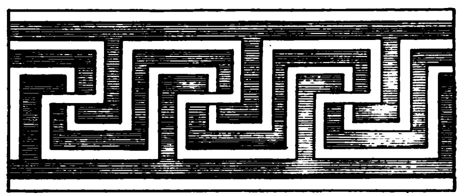
2. Greek Geometry. The early geometry, however valuable it may have been for practical purposes, was deficient in that it consisted simply of a set of rules obtained after centuries of

experimenting. *How* to get a result seemed to be its important question. That this condition did not last indefinitely is due to the genius of the Greeks. Being a race of thinkers and poets, they wished to know *why* a certain result must follow under a given set of conditions. For many years their wise men studied in Egypt. Upon returning they aroused among their followers a great interest in the study of geometry. Many new truths were now discovered, which were gradually arranged in a system. In this way geometry became a science. After about three hundred years of persistent study the Greeks produced a great masterpiece in the form of Euclid's "Elements of Geometry" (300 B.C.). This was a textbook on geometry, divided into thirteen chapters or "books." It was accepted at once as the great authority on all geometric questions, and has retained much of its importance to this day.

3. Purpose of Geometry. Geometry, in the form given it by the Greeks, is no longer primarily concerned with such practical activities as surveying. Its main purpose is the discovery and classification of the most important properties of points, lines, surfaces, and solids, in their relation to each other.

Hence we must find out how the words "solid," "surface," "line," "point," are used in geometry.

4. Space, Solids, and Surfaces. Every intelligent being has a notion of what **space** is. The schoolroom represents a portion of space. If it were not bounded by walls, or surfaces, it would extend indefinitely. Therefore we see that a completely inclosed portion of space arises only when there exist bounding surfaces.



Any limited portion of space is a **geometric solid**. Solids are bounded by **surfaces**.

5. Lines and Points. Again, the walls of the schoolroom would extend indefinitely if they were not bounded. But they are bounded by their intersections, the edges of the room. These edges are **lines**. Finally, these edges, or lines, would extend indefinitely if they did not terminate each other by their intersections. The intersections of lines are **points**.

6. A surface may be considered by itself, without reference to a solid.

A line may be considered by itself, without reference to a surface.

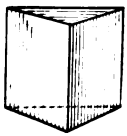
A point may be considered by itself, without reference to a line.

Surfaces are either **plane** (flat) or **curved.**

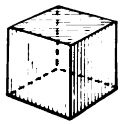
Lines are either **straight** or **curved.**

7. Examples of Solids. Geometric solids, surfaces, lines, and points are purely ideal objects. An ordinary solid, for instance, is material; but in geometry we are not concerned with the matter of which a solid is composed. We are interested only in its shape and size.

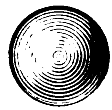
The following diagrams represent some of the most common solids:



PRISM

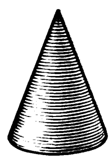


CUBE

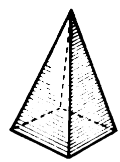


SPHERE

CYLINDER



CONE



PYRAMID

8. Value of Geometry. The study of geometry is of great value for two reasons:

In the *first* place, some knowledge of geometric principles is indispensable to any person desiring more than a superficial acquaintance with engineering, architecture, designing, drafting, physics, astronomy, surveying, or navigation. Of course it is impossible in an elementary book to point out in detail the connection between geometry and applied science, but the fact that it exists would be a sufficient reason for the appearance of geometry in every high-school course.

In the *second* place, there is no other subject that illustrates more clearly, when correctly taught, what it means to *prove* a statement, or emphasizes more strongly the necessity of accuracy in expression. Geometry also greatly increases one's power of mathematical reasoning, secures a deeper insight into spatial relations, and gives greater skill in classification. If to this be added the fact that the study of geometry requires no special mental equipment beyond ordinary common sense, and that geometric truths once established hold for all time, no further explanation is needed for the interest shown by many great thinkers in geometric investigations, nor for the prevailing opinion that every educated person ought to know something of geometric methods.

9. Method of Geometry. The history of geometry shows that whenever the subject was studied merely for practical purposes few important discoveries were made. The validity of a principle was determined by experiment, and people were satisfied if they found *how* to get a certain result. Progress began as soon as the attempt was made to give *reasons* for processes and rules. At the present time geometry can be studied either by experimenting or by reasoning. Either method is valuable, but it requires little reflection to see that a person who knows *why* he proceeds in a certain way is on safer ground than one who follows a rule blindly. The second method is, however, the more difficult one. In order to lessen the difficulty this book has been divided into two parts, an informal part and a formal part. The introductory course will lead by slow degrees from one method to the other. At first simple observational tests will be used; later a reason will be required for every statement.

PART I

THE STRAIGHT LINE

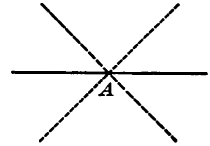
10. General Properties. The **straight line** and the **plane** are among the most fundamental concepts of geometry. For that reason it is difficult, if not impossible, to define them. The following simple tests may take the place of definitions:

1. Lay your pencil on the cover of your book. The pencil may be moved about freely on the surface of the cover, however you happen to place it. Could this be done on a curved surface?
2. Fold a sheet of paper. The crease represents a straight line.
3. If an elastic band or a piece of thread is stretched, it assumes the appearance of a straight line.
4. Look along the edge of your ruler. If the edge is straight, it can take on the appearance of a point.

EXERCISES

1. Locate on your paper, near the top, any convenient point. Represent this point by a pencil dot. Name the dot A.

2. Through this point draw a straight line. Observe that the drawing is not really a line, since lines are *ideal* objects, but it serves to *represent* a line.



3. How would you test the straightness of this line?

4. How could you make a straightedge from a sheet of paper?

5. Draw a dotted line through A; a spaced line (see figure).

6. How many straight lines can be drawn through a given point?

7. How many points can two straight lines have in common?

8. Explain why a point is sometimes represented by the symbol X.

9. Draw a straight line through the points A and B.

10. How many straight lines can be drawn through these two points?

11. Of what value is this fact in testing the straightness of a ruler? in testing a plane surface? in supporting a rod?

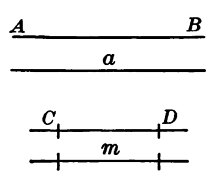
12. Into how many parts is a straight line divided by two points on it?

11. Summary. Properties of Straight Lines inferred directly:

1. *Through a given point an indefinite number of straight lines may be drawn.*
2. *Two straight lines can intersect in but one point.*
3. *Only one straight line can be drawn through two given points.*
4. *Two straight lines that have two points in common coincide throughout and form but one line.*
5. *Two straight lines do not inclose a space.*

12. Line-Segments. Measurement of Segments. Any drawing of a straight line in reality represents but a small portion of that line. Our imagination, however, can extend this limited portion indefinitely. It is sometimes convenient to distinguish between a limited and an unlimited line.

Any portion of a straight line lying between two of its points is called a **segment** of that line.



A straight line may be designated by naming two of its points, as the line AB; or by a single (small) letter, as the line a.

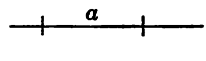
A segment is designated by naming its end points (extremities), as the segment CD; or by a single (small) letter, as the segment m.

EXERCISES

1. How many line-segments are there in the capital letter M? Indicate each segment by naming its extremities.

2. How would you test the equality or inequality of two segments?

3. Draw two straight lines. Indicate on the first an arbitrary segment a. On the second line mark off a segment equal to a by using (1) a strip of paper having a straight edge; (2) a ruler; (3) a pair of dividers. Which of these methods is the most convenient?



4. Draw five segments of different lengths. How can you compare one of them with a given length on the scale? Why do we use a **standard** scale? Name the principal standards of length.

5. Measure the five segments in inches; in centimeters. Tabulate your readings as follows:

Segment	Inches	Centimeters
a		
b		
c		
d		
e		

6. Draw segments of the following lengths: 2 in., $2\frac{1}{2}$ in., $5\frac{7}{8}$ in., $4\frac{3}{16}$ in., 12 cm., 8.8 cm.

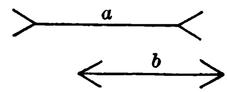
7. Draw a segment 10 cm. long and measure its length in inches. How many inches are there in a centimeter? Test your result by taking off, with the dividers, one centimeter on the centimeter scale and applying it to the inch scale.

8. Draw a segment 5 in. long and find its length in centimeters. From this calculate the number of centimeters in 1 inch.

9. Mark off on a line ten consecutive divisions of your centimeter scale. How can you test the accuracy of the scale?

10. Draw on your paper, or on the blackboard, several segments. Estimate their lengths and test by measuring them.

11. Estimate the comparative lengths of the segments *a* and *b* in the figure, and then test your estimate by means of the dividers.



12. On a map measure in inches or centimeters the distances between four cities chosen at random. With the aid of the scale given on the map express these distances in miles or kilometers.

13. Can you measure with absolute accuracy? If not, why are different degrees of accuracy possible?

13. Summary. 1. A segment is **measured** by comparing it with given standards of length. Measurements are only approximate. The degree of accuracy obtained will depend chiefly on the size of the unit chosen; that is, the smaller the unit, the more correct will be the measurement.

2. *If the extremities of a segment are given, the entire segment is thereby determined both as to position and length.*

3. The segment joining two points is known as the **measure of the distance** between the points.

4. *Two segments whose extremities can be made to coincide must coincide throughout.*

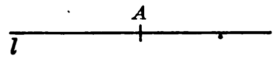
5. *Two segments a and b must be in one of three relations to each other:*

$$a > b \text{ (}a\text{ is greater than }b\text{)},$$
$$a = b \text{ (}a\text{ equals }b\text{)},$$
$$a < b \text{ (}a\text{ is less than }b\text{)}.$$

14. Opposite Directions. The Fundamental Operations. A satisfactory understanding of some new terms and processes may best be gained by solving the following exercises:

EXERCISES

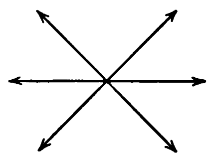
1. Given a line l and a point A on it. Required to change the position of A on l by 2 cm. How many solutions are possible? On your drawing denote the new positions of A by A_1 and A_2 (read "A one" and "A two").



2. How many directions can be distinguished on a straight line from a given point on that line?

3. Prolong a segment $AB = 4$ cm. by 3 cm. in the direction AB; in the direction BA.

4. How many directions are indicated by two straight lines crossing each other? by three lines passing through one point? four lines? n lines?



5. Draw a segment about 6 in. long and on it take off in succession $AB = 2$ in., $BC = 1\frac{3}{4}$ in., $CD = 1\frac{5}{8}$ in. Find the length of AD by computation and by measurement.

6. A man walks 4.5 mi. due north, and then $3\frac{5}{8}$ mi. due south. How far is he from the starting point? Draw a diagram of his route, 1 mi. being represented by 1 in.

7. A boat sails due east for 3 hr., at the rate of 7 mi. an hour. It then returns over the same course at the rate of 6.5 mi. an hour. Ascertain by means of a drawing how far the boat is from the starting point 2 hr. after turning about. (Represent 1 mi. by 1 cm.)

8. On a straight line lay off, four times in succession, a segment 2.5 cm. long. Measure the resulting segment. Test the result by calculation. How might the dividers be used to advantage in this exercise?

9. Divide a line 15 cm. long into five equal parts by inspection; by measurement.

10. On a straight line locate in order the points, A, B, C, D, E, F. Express

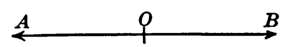
AC, BD, CF, each as the sum of two segments;
AD, BE, each as the sum of three segments;
AB, BD, each as the difference of two segments;
$AB + BD$, $BD + DF$, $AB + BD + DE$, each as a single segment.

11. A, B, C, are points on a line. If $AB = BC$, how is B situated?

12. Draw a line-segment and locate its middle point by inspection. Test by measurement.

15. **Summary**. The preceding exercises justify the statements of this section and the two following.

Any point of a straight line divides it into two parts, which are said to lie on **opposite sides** of that point, or to have **opposite directions**. Each of these parts is sometimes called a **ray**.



The point from which a ray extends is called its **origin**. Thus, in the figure, OA and OB are rays having the common origin O.

16. *Segments may be added together. Of two unequal segments the smaller may be subtracted from the larger. Segments may be multiplied or divided * by a number.*

17. If on a line three points, A, B, C, are taken in succession, so that $AB = BC$, B is called the **mid-point** of segment AC, or B is said to **bisect** segment AC, and A and C are **equally distant** from B.

18. **Equality of Segments**. A number of fundamental algebraic processes are also of great importance in geometry, as the following exercises will suggest.

* Division, at this point, is possible only if the given segment can be measured.

EXERCISES

1. Given a segment a. Draw a segment equal to $3a$.

2. Given two segments a and b. Draw segments equal to $a + b$, $a - b$, $3(a - b)$, taking $a > b$.

3. In Fig. 1,*

$$\begin{array}{ll} & a = a', \\ \text{and} & \underline{b = b'.} \\ \text{Explain why} & a + b = a' + b'. \end{array}$$

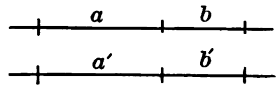


FIG. 1*

4. If, in Fig. 2, $AC = A'C'$, and $AB = A'B'$, explain why $BC = B'C'$.

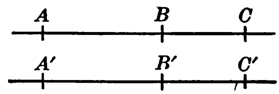


FIG. 2

5. In Fig. 3, $AB = CD$. Find two other equal segments.

FIG. 3

6. Given $AB = 3a$, and $CD = 3b$ (Fig. 4). If $a = b$, explain why $AB = CD$.

7. In the same figure

$$a = \frac{AB}{3},\ b = \frac{CD}{3}.$$

If $AB = CD$, explain why $a = b$.

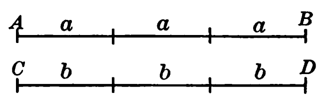


FIG. 4

8. Express in words the principles underlying Exs. 3–7.

REVIEW EXERCISES

1. Locate on a straight line, successively, one point; two points; three points; n points. How many segments arise in each case? How many rays?

2. What is the largest number of points in which three lines can intersect? four lines? five lines? n lines? Arrange your answers in the form of a table.

3. What is the largest number of lines determined (fixed) by two points? three points? n points?

* a' is read "a prime."

4. Given the segments $p = 28$ mm., $q = 16$ mm., $r = 9$ mm., $s =$ 37 mm. Draw segments equal to $p + q$; $s + r$; $s - q$; $p - r$; $p + r + q$; $p + q - s$.

5. Given $a = 15$ cm., $b = 6$ cm. Draw segments equal to $\frac{a+b}{2}$; $a - \frac{b}{3}$; $\frac{2a}{3} + \frac{3b}{4}$.

6. A segment 12 cm. long is to be divided into three parts which are to be in the ratio of 1, 2, 3. Find their lengths.

Solution. $1 + 2 + 3 = 6.$
$12 \div 6 = 2.$
$1 \times 2 = 2;\ 2 \times 2 = 4;\ 3 \times 2 = 6.$

Check. $2 + 4 + 6 = 12.$

7. The sum of the length and the width of a house is 70 ft. The width is to the length as 3 is to 4. What are the dimensions of the house?

8. A segment x units long is to be divided into five parts which are to be in the ratio of 1, 2, 3, 4, 5. Determine the length of each part.

9. A segment 18 in. long is divided into three parts such that the first is equal to the sum of the other two, while the sum of the first two is eight times the third. Find the length of each part.

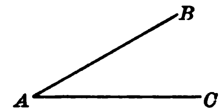
10. From a fixed point draw five rays. Lay off on these rays, from the common origin, the segments a, b, c, d, e, respectively, making $a = 2$ cm., $b = 3$ cm., $c = 4$ cm., $d = 5$ cm., $e = 6$ cm. Join the extremities of these segments in order, naming the new segments m, n, p, q. Measure m, n, p, q, and tabulate your results as follows:

Segment	Estimated length	Measured length	Actual error	Percentage error
m				
n				
p				
q				

11. Will the values of m, n, p, and q be the same in each case for various figures drawn in accordance with the above directions?

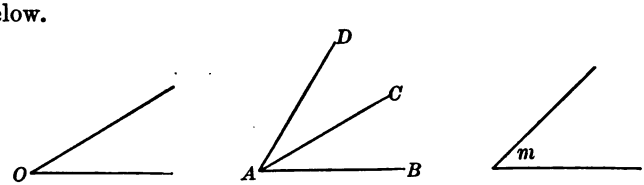
THE ANGLE

19. Definition and Notation. An **angle** is a figure formed by two rays having a common origin. The two rays, AB and AC, are called the **sides**, and A, their origin, is called the **vertex**, of the angle.



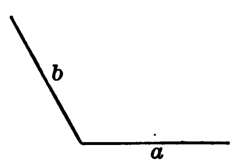
The outstretched fingers, the hands of a clock, the spokes of a wheel, the divisions of the compass, the gable of a roof, and many similar illustrations readily suggest the frequent occurrence and great importance of angles.

The methods of naming angles are shown in the figures below.



A single angle at a point may be designated by a capital letter placed at its vertex, as angle O.

When several angles have the same vertex, each angle may be designated by three letters; namely, by one letter on each of its sides, together with one at its vertex. The letter at the vertex is read between the other two, as angle DAC, angle CAB, etc.



Sometimes an angle is denoted by a small letter placed between its sides near the vertex, as angle m.

The sides may be denoted by small letters. The angle ab is the angle formed by the rays a and b. This notation is, however, used less frequently than the preceding.

The word "angle" is frequently replaced by the symbol $\angle$. The symbol $\angle$s is used as an abbreviation for "angles."

EXERCISES

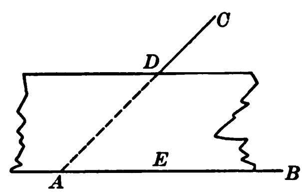
1. How many angles may be found in the letters N and W?

2. From a given point draw three rays; four rays; five rays. How many angles are formed in each case?

3. Through a given point draw two lines; three lines; four lines. How many angles are formed in each case?

4. Draw two angles, making them as nearly equal as you can. Test their equality by copying one of the angles on tracing paper, or by cutting out one of the angles, and applying it to the other.

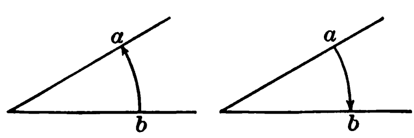
5. The figure shows how a strip of paper having a straight edge may be used to copy an angle. Place the straight edge on the side AB of $\angle BAC$, and mark on it the position of the vertex A. Mark also the point D, where the side AC intersects the other edge of the paper. Then $\angle DAE = \angle CAB$.



With the aid of this paper strip draw an angle equal to $\angle CAB$.

6. If you extend the sides of $\angle CAB$ beyond C and B, do you change the size of the angle?

7. Draw a ray and place your pencil on it. Revolve the pencil on the flat surface of the paper, using one extremity of the pencil as a pivot. Observe that each position of the pencil indicates a different angle. The pencil will eventually return to its first position. This rotation evidently gives a picture of every possible angle.



8. Show that in the figures a line could revolve from the position a to the position b, or from b to a.

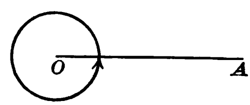
This fact is indicated by the arrowheads.

Classification of Angles

(A) The Three Fundamental Angles

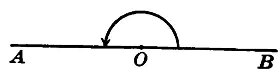
20. From the preceding exercises may be derived a general idea of the magnitude of an angle. It appears that the **magnitude of an angle** depends on the amount of revolution necessary to turn a line through the angle about the vertex as a pivot. The revolving line is said to **generate** or **describe** the angle. The size of an angle is independent of the length of its sides.

21. A **round angle, or perigon,** is an angle whose magnitude is indicated by a complete revolution of the generating line.



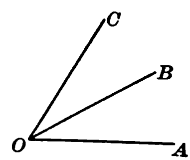
22. *All round angles are equal.*

23. A **straight angle** is an angle whose sides are in the same straight line, on opposite sides of the vertex; as $\angle AOB$.



24. *All straight angles are equal.* Why?

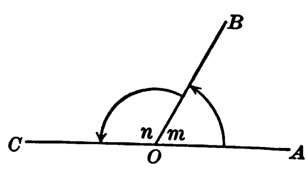
25. Two angles having the same vertex and a common side between them are called **adjacent angles**; as the $\angle s$ AOB and BOC.



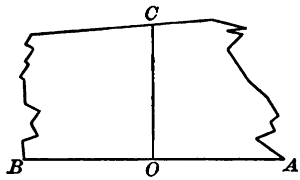
EXERCISES

1. Can two angles have a common side and a common vertex without being adjacent?

2. In the annexed figure $\angle AOC$ is a straight angle. If a ray OB revolves from the position OA toward the position OC, two angles are formed, $\angle m$ and $\angle n$. At first $\angle m$ is less than $\angle n$. Finally, however, $\angle m$ will be greater than $\angle n$. Hence *one* position of OB must have made the angles equal. Indicate approximately this position of OB in the figure.



3. Take a piece of paper having a straight edge *AB* and fold it so as to bring the point *B* on the point *A*. Open the paper and name the crease *OC*. Then $\angle AOC = \angle BOC$.



26. A **right angle** is an angle such that the adjacent angle formed by extending one of its sides beyond the vertex is equal to it. Thus, in the figure, $\angle BOC$ is a right angle.

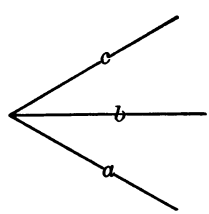
27. *A right angle is half of a straight angle.* Why?

28. *All right angles are equal.* Why?

29. The sides of a right angle are said to be **perpendicular** to each other; thus *OC* in the figure is perpendicular to *OB*, and *OB* is perpendicular to *OC*. This is sometimes abbreviated as follows: $OC \perp OB$, $OB \perp OC$. If *OC* is perpendicular to *AB*, *O* is called the **foot** of the perpendicular *OC*.

30. Round angles, straight angles, and right angles are so important that they are taken as **standards** with which other angles are compared.

31. If three rays, *a*, *b*, and *c*, are drawn from the same point, as shown in the accompanying figure, the following relations exist:



1. $\angle ac$ is the *sum* of $\angle ab$ and $\angle bc$.
2. $\angle ab$ is the *difference* between $\angle ac$ and $\angle bc$.
3. $\angle ac$ is *greater* than either $\angle ab$ or $\angle bc$.
4. If $\angle ab = \angle bc$, the ray *b* is the **bisector** of $\angle ac$.

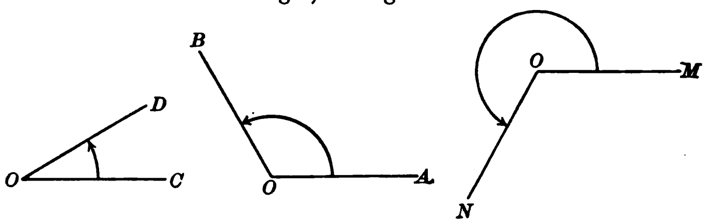
32. *Two angles A and B must be in one of the following three relations to each other:* $\angle A > \angle B$, $\angle A = \angle B$, or $\angle A < \angle B$.

(B) THE THREE OBLIQUE ANGLES

33. An **acute angle** is an angle less than a right angle, as the angle *COD*.

34. An **obtuse angle** is an angle greater than a right angle but less than a straight angle, as angle *AOB*.

35. A **reflex angle** is an angle greater than a straight angle but less than a round angle, as angle *MON*.



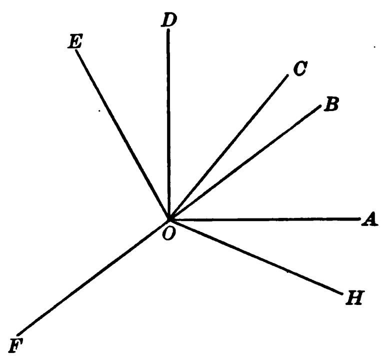
36. Acute, obtuse, and reflex angles are sometimes called **oblique angles**, and intersecting lines not mutually perpendicular are said to be **oblique** to each other.

EXERCISES

1. What kind of angle is each of the following angles of the annexed figure: $\angle AOB$, $\angle AOD$, $\angle AOE$, $\angle AOF$, $\angle BOD$, $\angle BOE$, $\angle BOF$, $\angle EOH$?

2. In the same figure what is the relation of *OA* to *OD*? *OC* to *OD*? *OB* to *OE*? *OB* to *OF*?

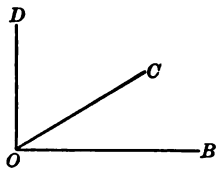
3. Draw an acute angle; a right angle (using a paper pattern obtained by folding); an obtuse angle.

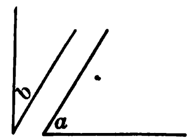


4. Point out right angles in the construction and equipment of the schoolroom; in the exterior of a house; in the street.

(C) THE FOUR SPECIALLY RELATED ANGLE-PAIRS

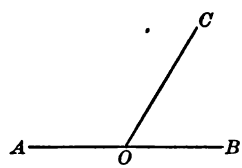
37. Two angles whose sum is equal to a right angle are said to be **complementary.** Each of the angles is called the **complement** of the other.





Thus ∠*BOC* is the complement of ∠*COD*; also ∠*a* is the complement of ∠*b*.

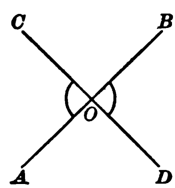
38. Two angles whose sum is equal to a straight angle are said to be **supplementary,** each being the **supplement** of the other.



In the accompanying figure ∠s *AOC* and *BOC* are supplementary.

39. Two angles whose sum is a round angle are said to be **conjugate** angles.

40. If two lines, *AB* and *CD*, intersect at *O*, as in the figure, the ∠s *AOC* and *BOD* are called **vertical** or **opposite** angles; also the ∠s *AOD* and *BOC*.



EXERCISES

1. Draw an acute angle, and then (using a pattern of a right angle) draw its adjacent complement. This may be done in two ways. Explain.

2. Draw an angle and construct its adjacent supplement. This may be done in two ways. Explain.

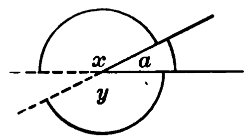
3. Draw the supplement and the complement of a given angle. Show that the difference of these two angles is a right angle.

4. If two angles are equal, how do their complements compare? their supplements? Why?

5. What kind of angle is equal to its supplement? less than its supplement? greater than its supplement?

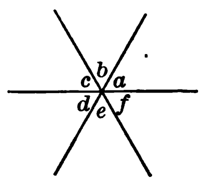
6. If one of two rods is made to turn about a point in the other as a pivot, two pairs of vertical angles are formed. As the vertical angles are generated by the same amount of turning, how do they compare in size?

7. Show the equality of two vertical angles by copying one of them on tracing paper.



8. Given an angle a. Construct its adjacent supplement in two ways. Does this suggest another reason for the *equality of vertical angles?*

9. In the accompanying figure find the vertical angle of $\angle a + \angle b$; the supplement of $\angle b$; the supplement of $\angle b + \angle d$. How many pairs of vertical angles and of supplementary adjacent angles in the figure? What is the sum of $\angle a$, $\angle c$, and $\angle e$?



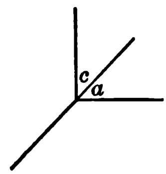
10. Prove that the vertical angles of two complementary angles are also complementary, and that the vertical angles of two supplementary angles are supplementary.

11. State the conclusions obtained in each of the following cases:

1. $\angle a$ is the supplement of $\angle b$, and
   $\angle b$ is the supplement of $\angle c$.
2. $\angle a$ is the complement of $\angle b$, and
   $\angle b$ is the complement of $\angle c$.
3. $\angle a$ is the supplement of $\angle b$, and
   $\angle b$ is the complement of $\angle c$.
4. $\angle a$ is the supplement of $\angle b$, $\angle b = \angle c$, and
   $\angle c$ is the supplement of $\angle d$.

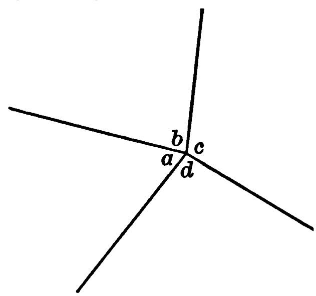
5. The supplement of $\angle a$ equals the supplement of $\angle b$.

6. The supplement of $\angle a$ is greater than the supplement of $\angle b$.



12. Show that the supplements of two complementary angles are together equal to three right angles.

13. Through the vertex of a right angle draw a straight line falling without the angle. Prove that the two acute angles formed are complementary.



14. In the figure $\angle a$ is the supplement of $\angle c$. Prove that $\angle b$ is the supplement of $\angle d$.

15. If two angles are complementary, what is the relation of their complements? If two angles are supplementary, what is the relation of their supplements?

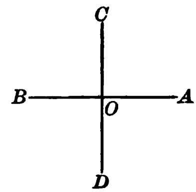
41. Summary. The truth of the following statements is now apparent:

1. *All round angles are equal.*
2. *All straight angles are equal.*
3. *All right angles are equal.*
4. *At a given point in a given line only one perpendicular can be drawn to that line (in the same plane).*
5. *The complements of the same angle, or of equal angles, are equal.*
6. *The supplements of the same angle, or of equal angles, are equal.*
7. *Vertical angles are equal.*
8. *If two adjacent angles have their exterior sides in a straight line, they are supplementary.*
9. *If two adjacent angles are supplementary, their exterior sides lie in a straight line.*

Measurement of Angles

42. From the preceding study of angles it is evident that a round angle equals two straight angles or four right angles.

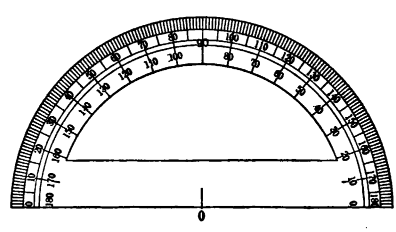
Fold a piece of paper of any shape and call the straight folded edge AB. Fold again so as to bring B on A. Now open the paper. The four angles formed by the creases are right angles. Why? By continuing to fold the paper properly we may obtain eight equal angles, sixteen equal angles, etc. This is a mechanical way of dividing the round angle into 4, 8, 16, etc., equal parts.



43. In measuring angles we imagine the angular magnitude about a point in a plane to be divided into 360 equal parts, called **degrees**, and we simply state how many degrees the angle under consideration contains. Thus the degree is $\frac{1}{360}$ of a round angle, $\frac{1}{180}$ of a straight angle, $\frac{1}{90}$ of a right angle.

The degree is again divided into 60 equal parts called **minutes**, and the minute into 60 equal parts called **seconds.** The expression 40 degrees 7 minutes 20 seconds is written $40° 7' 20''$.

44. A special instrument, called a **protractor,** is frequently used for angle measurements. The figure represents one form of the protractor. By joining the notch O of the protractor to each graduation mark we obtain a set of angles at O, each usually representing an angle of one degree.



To measure a given angle with the protractor, place the notch of the protractor at the vertex of the angle, and the base line along one side of the angle. The other side of the angle then indicates on the protractor the number of degrees in the angle.

To draw an angle of a given number of degrees, place the base of the protractor along a straight line and mark on the line the position of the notch O. Then place the pencil at the required graduation mark and (after removing the protractor) join the point so marked to O.

EXERCISES

1. Divide a straight angle into three parts, a, b, c. Estimate the size of these angles. Then measure each angle with the protractor and tabulate your results as follows:

ANGLE	DEGREES		ERROR
	Estimated	Measured	
a			
b			
c			
Sum			

2. Divide a round angle into five parts and proceed as in Ex. 1.

3. With a protractor draw angles of 20°, 30°, 45°, 90°, 36°, 120°, 135°, 225°.

4. How many degrees in the following fractions of a right angle: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{15}$, $\frac{1}{18}$, $\frac{1}{30}$, $\frac{2}{30}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{3}{4}$, $\frac{3}{10}$?

5. How many degrees in the following fractions of a straight angle: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{12}$, $\frac{1}{15}$, $\frac{1}{18}$, $\frac{1}{20}$, $\frac{1}{30}$, $\frac{1}{36}$, $\frac{1}{45}$, $\frac{1}{90}$?

6. Which integers are exactly contained in 360? Does this suggest why a round angle is divided into 360 parts?

7. How many degrees in the complement of each of the following angles: 20°, 30°, 45°, 67°, 89°, $a°$, $(a + b)°$, $(45 + a)°$?

8. How many degrees in the supplement of each of the following angles: 60°, 45°, 90°, 110°, $x°$, $(x - a)°$?

9. An angle is $\frac{1}{2}$ ($\frac{1}{3}$, $\frac{1}{4}$) * of its supplement. How many degrees in the angle?

10. An angle is 3 (2, 4, 5) * times as great as its complement. How many degrees does it contain?

11. The supplement of an angle is three times its complement. How many degrees in the angle?

Suggestion. $180 - x = 3\ (90 - x)$.

* Each of the values given in the parenthesis is to be substituted in place of the value given before the parenthesis, making a new exercise.

12. What fractions of a right angle are the angles formed by the hands of a clock, from hour to hour, between 12 and 6 o'clock? What fractions of a straight angle? How many degrees in each angle? State in each case whether the angle is acute, right, or obtuse.

13. How many degrees in the angles formed by the hands of a clock when the time is 3.12, 11.48, 10.48, 8.24, 9.36, 6.20, 7.22 (railroad notation)? (Remember that the minute hand travels 12 times as fast as the hour hand, and that a minute division on the face of the clock represents 6°.)

14. In how many minutes does the minute hand revolve through an angle of 90°? 45°? 270°? In how many hours does the hour hand describe these angles?

15. What angle is described in an hour by the minute hand? by the hour hand?

16. A man, setting his watch, moves the minute hand forward half an hour and then moves it back 8 min. How many degrees in the angle between the first and the final position of the minute hand?

17. The driving wheel of an engine revolves 10 (20, 30, x) * times per minute. In what time does one of the spokes turn through a right angle?

18. A screw required $10\frac{1}{2}$ complete turns before it was firm in the wood. The depth of the hole was found to be $\frac{3}{4}$ in. How far did a turn of a straight angle drive it?

19. What is the complement of 24° 17′? of 79° 11′? of 46° 34′ 10″?

20. Find the sum of the following angles: 26° 47′ 3″, 44° 22′ 32″, 68° 51′ 48″, 39° 58′ 37″.

21. Find the supplements of the following angles: 163° 17′, 48° 34′, 94° 52′ 21″.

22. The supplement of an angle is 8 times its complement. Find the value of the angle in degrees, minutes, and seconds.

23. Reduce to degrees, minutes, and seconds $\frac{7}{22}$ of a straight angle.

* Each of the values given in the parenthesis is to be substituted in place of the value given before the parenthesis, making a new exercise.

24. The accompanying figure shows the card of the mariner's compass. Count the number of equal parts into which the "points of the compass" divide the round angle. How many degrees in each of these parts? Draw a compass card.

25. Determine by how many degrees the following directions differ:

N. and E.	W. and N.E.
N. and W.	N.N.E. and E.S.E.
N. and S.	N.W. and S.E.
S. and S.W.	N.W. and S.W.
N. and E.S.E.	S.S.E. and W.N.W.
W.S.W. and S.E.	S.W. by S. and S.W. by W.
S. and N.W.	N.W. by W. and S. by E.

26. Determine the course of a ship, that is, the point of the compass toward which a ship is sailing, in each of the following cases:

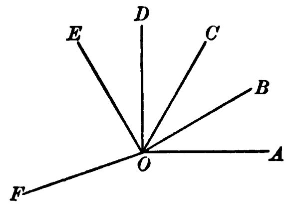
N. 45° W.	S. 45° E.	N. $67\frac{1}{2}$° W.	S. $22\frac{1}{2}$° E.
S. $11\frac{1}{4}$° W.	N. 90° E.	S. $67\frac{1}{2}$° E.	S. $56\frac{1}{4}$° W.

NOTE. N. 45° E. means that the ship is sailing northeast. It signifies a direction 45° east of north.

EXERCISES

Review and Extension

1. From a point O as a common origin draw six rays indicated in order by the letters A, B, C, D, E, F. Express $\angle AOD$ and $\angle BOF$ each as the sum of three angles; $\angle AOC$ and $\angle BOD$ each as the difference of two angles; $\angle AOB + \angle BOD - \angle COD$ as a single angle.



2. If $\angle AOB = \angle BOC$, what name is given to OB?

3. In the above figure,

if $\angle AOB = \angle COD$, show that $\angle AOC = \angle BOD$;
if $\angle AOC = \angle BOD$, show that $\angle AOB = \angle COD$;
if $\angle AOB = \angle COD$, while $\angle AOD = 3\angle AOB$ and $\angle BOE = 3\angle COD$, show that $\angle AOD = \angle BOE$;
if $\angle AOB = \frac{1}{2}\angle AOC$, and $\angle COD = \frac{1}{2}\angle COE$, while $\angle AOC = \angle COE$, show that $\angle AOB = \angle COD$.

Express in words the principles underlying these relations.

4. On one side of a straight line, and having a common vertex, are situated (a) three (four, five) equal angles; (b) four angles, of which each after the first is twice as large as the preceding one. How many degrees in each angle?

Let $x =$ number of degrees in the first angle.

5. There are four angles about a point, of which each after the first is three times as large as the preceding angle. How many degrees in each angle?

6. What is the result in Ex. 5, if the second angle is three times the first, and the third and fourth are each equal to the sum of the two preceding angles?

7. One of two supplementary angles is 2 (3, 4) times as large as the other. How many degrees in each angle?

8. There are five angles about a point. Four of them have the following magnitudes: 74°, 101°, 59°, 94°. How many degrees in the fifth angle?

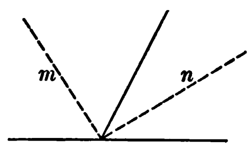
9. Given $\angle a = 40°$, $\angle b = 30°$, $\angle c = 20°$, $\angle d = 10°$. Draw angles equal to $\angle a + \angle b$, $\angle b + \angle d$, $\angle b - \angle c$, $\angle b + \angle a - \angle d$.

10. Given $\angle a = 150°$, $\angle b = 60°$. Draw angles equal to $\frac{\angle a + \angle b}{2}$; $\angle a - \angle b$; $\angle a + \frac{\angle b}{3}$; $\frac{2}{3}\angle a + \frac{3}{4}\angle b$.

11. How many degrees does the minute hand of a clock traverse in 1 hr.? in $\frac{1}{2}$ ($\frac{1}{4}$, $\frac{3}{4}$) hr.? in 30 (20, 10, 25) min. of time? in 5 (12, 24, 36) min. of time?

12. In what time does the minute hand of a clock describe an angle of 45°? 90°? 180°? 270°?

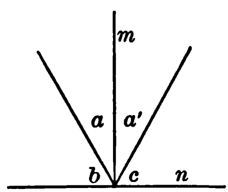
13. Change to the lowest indicated denominations $20° 24'$; $30° 30'$; $179° 59' 60''$.



14. Draw an angle of 50° and bisect it with the aid of the protractor.

15. In the figure m and n are the bisectors of the two supplementary-adjacent angles. Prove that $m \perp n$.

16. If an angle is bisected, prove that the supplements of the two equal angles are equal.



17. What kind of angle is formed by the bisectors of two vertical angles?

18. In the figure $m \perp n$, and $\angle a = \angle a'$. Show that $\angle b = \angle c$.

19. The sum of an acute angle and an obtuse angle cannot exceed how many degrees?

20. If $\angle a$ is greater than $\angle b$, show that the supplement of $\angle a$ must be less than the supplement of $\angle b$.

21. How many degrees in an angle which is 12° less than its supplement? 18° greater than its complement?

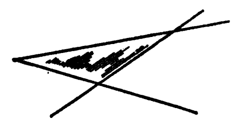
22. Find $\angle A$ and $\angle B$, if $\frac{1}{2}(\angle A + \angle B) = 48° \; 16' \; 20''$, while $\frac{1}{2}(\angle A - \angle B) = 22° \; 52' \; 17''$.

23. Find the value of one half the supplement of $65° \; 11' \; 31''$.

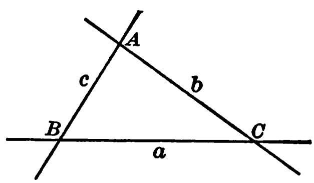
24. How many complete revolutions in an angular magnitude of 7832°? How many degrees between the initial and final positions of the revolving ray?

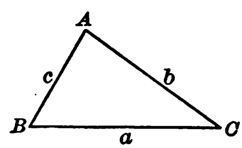
TRIANGLES

45. Experience teaches us that *two* straight lines do not inclose a space. If, however, the sides of an angle are crossed by a straight line, we obtain a closed space.



A **triangle** is a figure bounded by three straight lines. The points of intersection of the lines are called the **vertices,** and the segments joining the vertices are called the **sides** of the triangle.





The vertices are denoted by capital letters and the sides by small letters. Sometimes it is convenient to use corresponding letters, that is, the side opposite $\angle A$ is denoted by the corresponding small letter a.

The sum of the three sides of a triangle is called the **perimeter**.

46. The Angles of a Triangle. Exterior Angle. The following exercises are intended to develop a clearer insight into the mutual relations of the angles of a triangle.

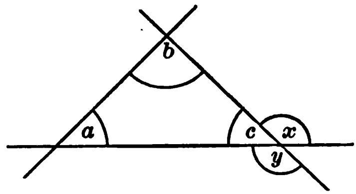
EXERCISES

1. Point out triangles in the exterior of a house; in the street; on the surfaces of certain solids (e.g. pyramids).

2. Draw five triangles of different shapes, and measure the sides of each. Tabulate your results. Can any three segments be used for the sides of a triangle?

3. Measure the angles of the triangles and tabulate. You will find that the sum of the three angles in each case is equal to 180°. It will be *proved* later that this is true of all triangles; that is, that the sum of the angles of *any* triangle equals a straight angle.

An **exterior angle** of a triangle is the angle formed by one side of the triangle and the prolongation of another side.



4. In the figure $\angle x$ is an exterior angle, while the angles a, b, c, are **interior angles**. Draw the figure.

5. How many exterior angles can a triangle have? Compare $\angle x$ and $\angle y$. How many pairs of vertical angles in the figure?

6. Draw these triangles, using the given measurements:

AB	*BC*	*CA*	$\angle A$	$\angle B$	$\angle C$
4 in.	5 in.			30°	
7 cm.		6 cm.	40°		
	40 mm.	70 mm.			80°
5 in.			45°	45°	
	5 cm.			90°	60°
		4 in.	60°		60°
	8 cm.			77°	46°
		2.8 cm.	48°		116°
7.3 cm.	12.1 cm.			28°	
		8 cm.	72°		72°
6.7 cm.	6.7 cm.			126°	
		2 in.	30°		60°
$3\frac{7}{8}$ in.	$2\frac{3}{8}$ in.			32°	

7. How many degrees in the angles of a triangle if the ratio of the angles is 1 : 2 : 3 ? 1 : 1 : 2 ? 2 : 3 : 5 ? 3 : 5 : 7 ?

8. How many right angles may a triangle have ? How many obtuse angles ?

9. How many acute angles *must* every triangle have ?

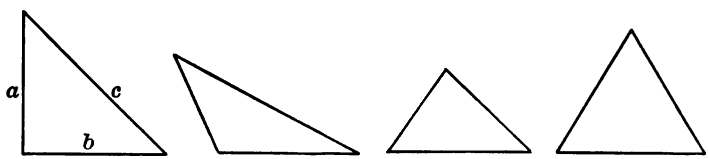
10. Can a triangle be drawn whose interior angles are 40°, 50°, 100° respectively ? 80°, 70°, 40° ? 90°, 40°, 50° ?

11. Two angles of a triangle are together equal to 100°. What is the value of the third angle ?

12. Verify by measurement the fact (to be proved later) that an exterior angle of a triangle is equal to the sum of the two remote interior angles.

Classification of Triangles based on Angles

47. A triangle, one of whose angles is a right angle, is called a **right triangle.** The sides inclosing the right angle are called the **legs,** and the side opposite the right angle is called the **hypotenuse.**



RIGHT OBTUSE ACUTE EQUIANGULAR

48. A triangle, one of whose angles is an obtuse angle, is called an **obtuse triangle.**

49. A triangle, each of whose angles is an acute angle, is called an **acute triangle.**

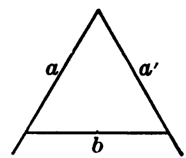
50. Acute triangles and obtuse triangles are called **oblique triangles.**

51. A triangle whose three angles are equal is called an **equiangular triangle.**

The word "triangle" is often replaced by the symbol △.

THE ISOSCELES TRIANGLE

52. On the sides of an angle lay off equal segments from the vertex and join their extremities. The result is a triangle having two equal sides. Such a triangle is said to be **isosceles.** The equal sides are called the **legs,** and the other side is the **base** of the triangle. In the figure b is the base.



The angle opposite the base of an isosceles triangle is called the **vertex angle** of the triangle.

The angles at the ends of the base of an isosceles triangle are called the **base angles** of the triangle.

EXERCISES

1. Draw an isosceles triangle whose vertex angle is acute; right; obtuse.

2. Draw three isosceles triangles. Measure the base angles in each. What do you observe?

3. Draw two intersecting lines and form isosceles triangles lying on opposite sides of the intersection point.

4. A plane divides space into two regions which lie on opposite sides of the plane. A line in a plane divides the plane into two regions such that the segment joining points on opposite sides of the line crosses that line. Into how many regions do the bounding lines (produced) of a triangle divide the plane?

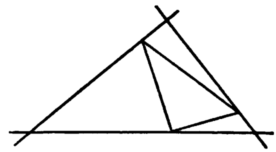
5. Give illustrations of isosceles triangles (for example, gable of a house, faces of a pyramid).

6. Show by a drawing that if an unlimited straight line intersects one side of a triangle at any point except an extremity, it must pass through the interior of the triangle and cut one of the other sides.

7. A line-segment is drawn from one vertex of a triangle to the opposite side, dividing the triangle into two triangles. If one of these triangles is acute, what is the other? If one is right, what is the other?

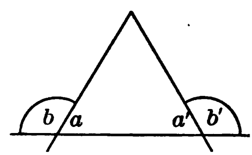
8. Through one vertex of a triangle draw a straight line falling without the triangle. What is the sum of the three angles formed at that vertex?

9. Through the three vertices of a triangle draw lines falling without the triangle, so that a new triangle is formed. This triangle is said to be **circumscribed** about the given triangle. The original triangle is said to be **inscribed** in the new triangle.



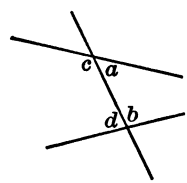
10. Lay off on a straight line lengths equal to the perimeters of the inscribed and of the circumscribed triangle. How do they compare?

11. If in the adjoining figure $\angle a = \angle a'$, prove that $\angle b = \angle b'$.



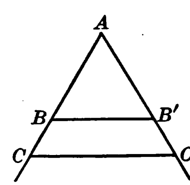
12. Produce two sides of a triangle beyond their point of intersection, making the extensions equal respectively to the sides. Join the extremities of the extensions. Show with tracing paper that the new triangle is a repetition of the first.

13. If in the adjoining figure $\angle a + \angle b$ is less than 2 rt. ∡, prove that $\angle c + \angle d$ is greater than 2 rt. ∡. On which side of the transverse line will the other two lines meet when produced?



14. A straight railroad track extending directly away in the line of vision appears to vanish in the distance at a single point. What geometrical figures do the rails and the ties *appear* to form?

15. If the triangles ABB' and ACC' in the adjoining figure are isosceles, what is the relation between BC and $B'C'$? Give a reason for your answer.



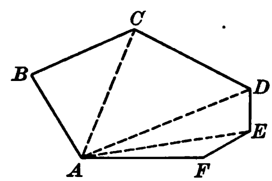
16. Draw five rays from a common origin. On each lay off three given segments, a, b, c, in succession, beginning at the origin. Connect the corresponding extremities. Of what does the figure remind you? How many isosceles triangles in the figure?

POLYGONS

53. A plane figure may be bounded by three sides, four sides, five sides, etc. The general name for such a figure is "polygon."

A **polygon** is a plane figure bounded by straight lines. The sides, the vertices, the interior and the exterior angles, and the perimeter of a polygon are defined as in the case of a triangle. Two angles at the extremities of the same side are said to **include** that side.

54. Polygons classified as to the Number of Sides. Polygons are named according to the number of their sides. A *quadrilateral* has four sides; a *pentagon,* five sides; a *hexagon,* six sides; an *octagon,* eight sides; a *decagon,* ten sides; a *dodecagon,* twelve sides, etc.



55. A **diagonal** of a polygon is a segment joining two vertices that are not consecutive. In the figure AC, AD, AE, are diagonals.

Unless the text contains a remark to the contrary, we shall use only polygons each of whose interior angles is less than a straight angle. Such polygons are called **convex** polygons.

56. Angles; Diagonals. The relations existing between the sides, the angles, and the diagonals of a polygon may be readily inferred from such exercises as the following:

EXERCISES

1. Draw a quadrilateral and produce the sides beyond the vertices. How many exterior angles are formed? How many pairs of vertical angles?

2. Answer the same questions for polygons of 5, 6, 7, 8, 9, 10, n sides.

3. How many diagonals can be drawn from one vertex of a polygon of 4 (5, 6, 7, 8, 9, 10, n) sides? Tabulate your answers.

4. In the previous exercise how many triangles are formed by these diagonals?

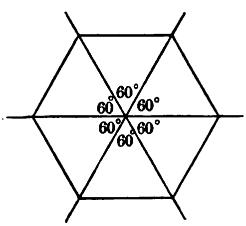
5. How many different diagonals can be drawn in a quadrilateral? in a pentagon? in a hexagon? Tabulate your answers.

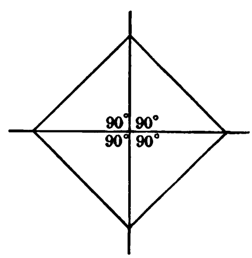
6. How many pairs of vertical angles do these diagonals form within the polygons?

7. Join any point within a polygon to all the vertices. How many triangles are formed?

8. Join any point, *not* a vertex, in the perimeter of a polygon to all the vertices. How many triangles are formed?

57. Regular Polygons. Since 60 is contained exactly in 360, a set of six angles, each equal to 60°, can be drawn around a point. If on the sides of these equal angles equal segments are laid off from the common vertex, and the points of division are joined in order, a polygon is formed. In the same manner angles of 90°, 72°, 45°, 36°, may be used. It will be proved later that the sides and the angles of such polygons are equal; for that reason they are called **regular polygons**.





EXERCISES

1. With the aid of the protractor draw regular polygons of 4, 5, 6, 8, 10 sides.

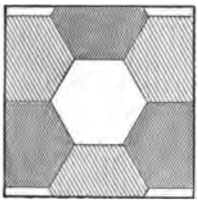
2. Test your drawings by measuring their sides and angles.

3. How many degrees in each of the interior angles of these polygons? in each of the exterior angles? Tabulate your answers.

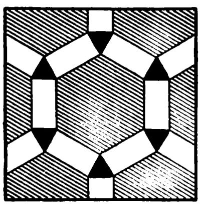
4. In which of these polygons do the rays form straight lines? Why?

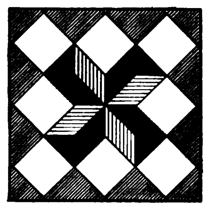
5. Give concrete illustrations of regular polygons.

6. With the aid of ruler, protractor, and compasses copy and extend the following patterns, which are frequently used in tiling or parquet flooring. Very artistic effects may be secured by coloring the parts of the figures in accordance with some definite plan.

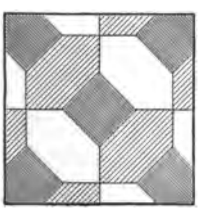


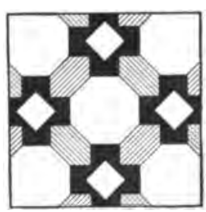
7. Draw the following patterns. In the first figure a regular hexagon is surrounded by rectangles and equilateral triangles.





8. In the following patterns irregular hexagons intervene between the regular polygons. Draw the figures.

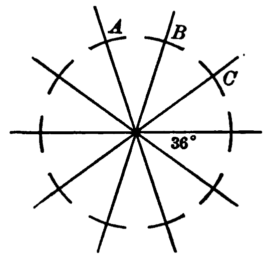




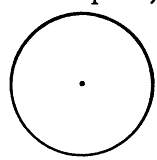
NOTE. Interesting applications of geometry are furnished by the field of ornamentation and design. It has been found that the earliest decorative patterns of nearly all primitive races are of a geometric character. These patterns usually grew out of such arts as weaving, the making of vases, or the laying of tile floors. The American Indians skillfully introduced simple designs in their basketry and pottery, and some Mexican rock temples show profuse geometric embellishment.

THE CIRCLE

58. Preliminary Definitions. A regular polygon of n sides may be obtained by drawing from a point n rays forming n equal angles, and laying off on these rays equal segments. The points of division (vertices) $A, B, C, \ldots$ could obviously have been obtained more rapidly by drawing a circle. The common origin is the **center,** and each of the segments a **radius** of the circle.



59. A **circle** is a closed curved line in a plane, all points of which are equally distant from a fixed point in the plane, called the *center*.



Many writers use the term "**circumference**" in the above sense, a circle being defined as a plane figure bounded by a circumference. There is good authority for our definition, which has the merit of agreeing with common usage and the language of higher mathematics.

The circle plays a very important part in geometry. It is the only curved line we shall have occasion to study in this text. Its use in many of the most fundamental constructions will prove its importance. Hundreds of concrete illustrations of the circle might be given here (e.g. ring, coins, dial of clock, wheel, protractor). We call attention especially to the "round" bodies — the sphere, the cylinder, the cone — from which circular sections may be obtained (e.g. sections of a ball, tree, megaphone).

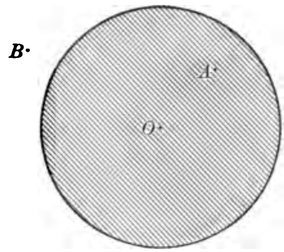
A line-segment connecting the center and any point of the circle is called a **radius.** Any line-segment passing through the center and having its extremities in the circle is called a **diameter.**

From these definitions it follows that:

60. *All radii of the same circle are equal. All diameters of the same circle are equal.*

61. *Two circles having equal radii or equal diameters can be made to coincide and are equal.*

62. A circle divides the plane into an **interior** and an **exterior** region. A point is in the interior or in the exterior region, according as the line joining it to the center is less than or greater than the radius. Any interior point A is said to be **within** the circle, while any exterior point B is **without** the circle.



63. A part of a circle (circumference) is called an **arc**. One half of a circle is called a **semicircle**. A fourth part of a circle is called a **quadrant**.

To draw a circle we use a special instrument, the compasses. A pencil or a crayon with a piece of string attached serves the same purpose.

EXERCISES

1. How would you locate on a map all houses that are 1 mi. from the City Hall?

2. What is the meaning of the statement, "The earthquake was felt within a radius of 100 mi."?

3. What kind of path does the extremity of a clock hand describe?

4. As you open your book observe the path described by a corner of the cover.

5. How does a gardener make a circular flower bed?

6. Revolve a circular piece of paper about one of its diameters. What kind of solid is generated?

7. If a wheel is made to slide along the axle passing through its hub, what kind of solid is generated?

8. A wheel revolves ten times per second. In what time does a point on its rim pass through a quadrant?

9. In what time does the extremity of the minute hand of a clock describe a semicircle?

10. A string fitting exactly around a bicycle wheel was found to be $7\frac{1}{3}$ ft. long. If the wheel makes three revolutions per second, what is the distance traversed in one minute?

Points and Circles

11. Show that a point may be in any one of *three* different positions with reference to a circle.

12. Given a point *P*. Find all the points that are 2 cm. from *P*.

13. Given a circle and a point *P*. Find the points on the circle that are 3 cm. from *P*. When is the solution impossible?

14. Given two points *A* and *B*. Find a point that is 3 cm. from *A* and 4 cm. from *B*. How many solutions are possible? When is the solution impossible?

15. Given a point. Required to draw a circle of radius 2 cm. that shall pass through this point. How many solutions are possible?

64. Lines and Circles. The different relative positions of a line and a circle are illustrated in the following exercises.

EXERCISES

1. Draw a circle and lay your pencil on the paper so that it passes through the center. Move the pencil away from the center. Observe that a line either cuts a circle in two points, or touches the circle, or falls entirely without the circle.

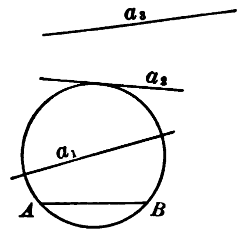
2. Given a line and a point *not on the line.* Draw three circles having the given point as center and illustrating the fact that there are *three* possible positions of a line with reference to a circle.

3. Given a line *l* and a point *P*. Required to find on *l* the points that are 5 cm. from *P*. When do you obtain two solutions? one solution? no solution?

SUMMARY

65. A **secant** is a straight line of unlimited length which intersects a circle.

66. A **tangent** is a straight line of unlimited length which has one and only one point in common with a circle. The point is called the **point of contact** or **point of tangency.**



67. A **chord** is a line-segment joining any two points of a circle.

In the figure a_1 is a secant, a_2 is a tangent, while AB is a chord.

68. *A secant to a circle intersects the circle in two and only two points.*

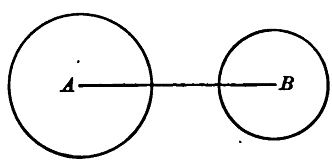
69. Two Circles. The different relative positions of two circles, and other geometrical properties, are developed in the following exercises:

EXERCISES

1. Cut out two circular pieces of paper, making their radii 3 cm. and 5 cm., and place them in each of the following positions: (1) one falls entirely without the other; (2) they touch externally; (3) they intersect; (4) they touch internally; (5) one is entirely within the other; (6) their centers coincide.

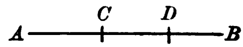
2. Draw two unequal circles in *six* different relative positions.

3. Draw two circles whose centers are A and B. Join A and B.



4. Draw a segment AB 3 in. long. On it indicate any two convenient points $\frac{3}{4}$ in. apart. Name these points C, D, so that the order of the points is A, C, D, B.

5. About A as a center, with AD as a radius, and also about B as a center, with BC as a radius, draw circles. How many points do these circles have in common?



6. Repeat Ex. 5, using A and B as centers, and taking for radii the segments AD and BD respectively. How many points are common to the two circles?

7. Repeat Ex. 5, using A and B as centers, and taking for radii AC and BD respectively. How many points are common to the circles?

8. Will the results of Exs. 5, 6, 7, be the same if the distances separating A, C, D, B, are changed, and only the *order* remains the same?

9. In each of Exs. 5–7 state the relation between the length of the segment joining the centers and the sum of the radii of the two circles.

10. If the lengths of the segment between the centers and of the radii of two circles be represented by l, r, r' respectively, express the relation between l, r, and r' in each of the six positions of two circles obtained in Ex. 2.

11. Draw five circles with the same center. Give concrete examples of such circles.

12. Can two circles which have the same center touch or intersect?

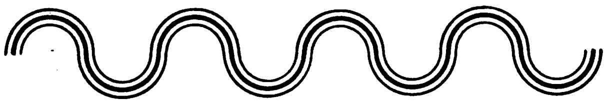
70. Center Line; Center Segment. The line passing through the centers of two circles is called their **line of centers** or **center line,** and the segment of the center line joining the centers is called their **center segment.**

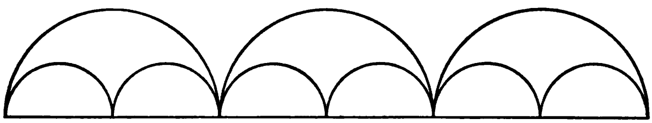
71. *If the center segment of two circles is less than the sum but greater than the difference of their radii, the circles intersect in two and only two points.*

72. Concentric Circles. Circles having the same center are said to be **concentric.**

EXERCISES

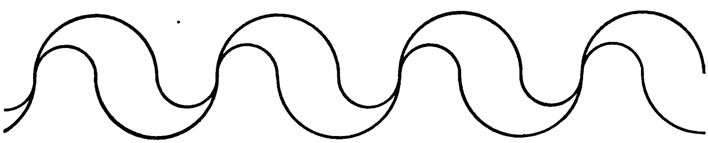
1. Make a careful copy of the two following figures:



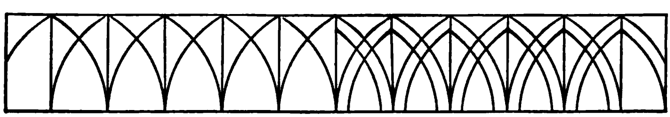


The figures are made up of semicircles having their centers in the same straight line.

2. Draw the following figure, the diameters of the smaller semicircles being half as large as those of the greater semicircles, and the centers of the semicircles being in the same straight line.

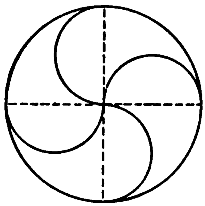


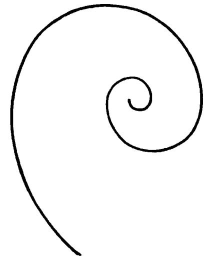
3. On a straight line lay off in succession a set of equal segments. With the extremities of these segments as centers, and with a radius equal to twice one of the segments, draw arcs of circles as shown in the figure. The smaller arcs are constructed from the same centers. Draw the figure and complete it.



4. Draw a circle of radius 3 cm. Using any point on this circle as a center, and with the same radius, draw another circle. With each of the two intersection points on the first circle as centers, keeping the same radius, draw circles. Repeat the process, using only points on the first circle as centers. The result is a design frequently used in ornamentation.

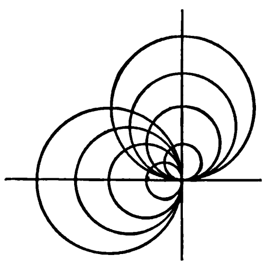
5. Draw the following figures:

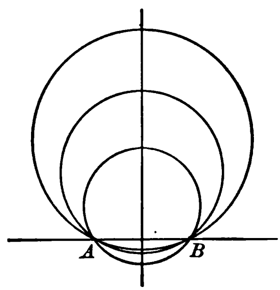


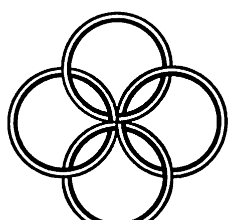


In the first figure the semicircles have as their diameters the radii of the outer circle. The second figure is a "spiral," composed of consecutive semicircles, each new radius being twice the previous radius and the centers lying on a straight line, each at the extremity of the arc preceding.

6. Copy and complete the following figures:

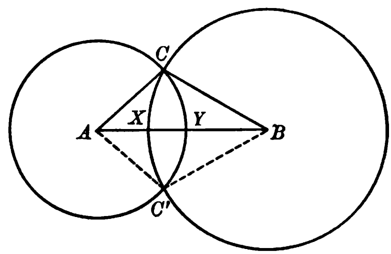






In each case draw two mutually perpendicular straight lines. In the first figure lay off on the four rays equal consecutive segments. Then draw four sets of circles, two of which are shown in the diagram.

73. Construction of Triangles, given the Sides. If the extremities of a segment AB are used as the centers of circles, it is obviously possible to find two radii, AY and BX, such that the two circles intersect. If either point of intersection is joined to A and B, a triangle is formed.



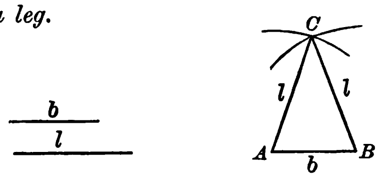
The circles will intersect when the line of centers is less than the sum of the radii; that is, when $AB < AC + BC$. We infer from this that *a triangle can be constructed with three given lines as sides, when the sum of any two sides is greater than the third side.*

74. Construction I. *To construct a triangle, given the three sides.*

The method of construction is shown in the figure.

75. Scalene Triangle. A triangle in which no two sides are equal is said to be **scalene.**

76. Construction II. *To construct an isosceles triangle, given the base and a leg.*



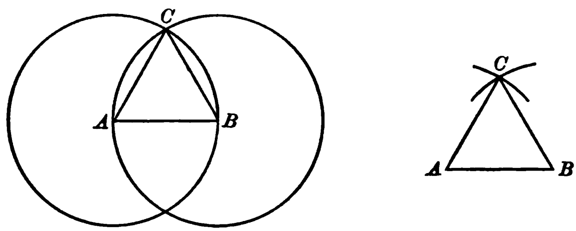
The method of construction is shown in the figure.

Can b and l be of any length?

77. The vertex C in the figure is said to be **equidistant** from A and B.

78. Equilateral Triangle. A triangle of which all three sides are equal is said to be **equilateral.**

79. Construction III. *To construct an equilateral triangle, given a side.*



$AC = AB = BC$. Hence the three sides of the triangle ABC are equal.

NOTE. This construction is the first proposition of Book I of Euclid's Elements.

EXERCISES

1. Construct triangles, when possible, from the following data, a, b, and c representing the sides. What kind of triangle results in each case?

a	b	c
4 cm.	5 cm.	6 cm.
3 in.	4 in.	5 in.
4 in.	5 in.	9 in.
7 cm.	8 cm.	8 cm.
5 cm.	5 cm.	8 cm.
3 in.	3 in.	3 in.
1 in.	2 in.	4 in.

2. Construct a point equidistant from two given points. How many solutions are possible?

3. Construct six different isosceles triangles on the same base. Join their vertices in succession. What do you observe?

4. Construct an equilateral triangle *ABC*, and on each of its sides construct an equilateral triangle. Show that if the three triangles surrounding triangle *ABC* be folded over on the sides of the original triangle, a pyramid can be made, having the given triangle *ABC* as its base.

5. Draw several circles so that each passes through two given points.

6. Can two circles intersect in three points?

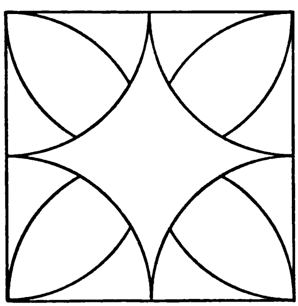
7. Draw three circles such that each intersects the other two.

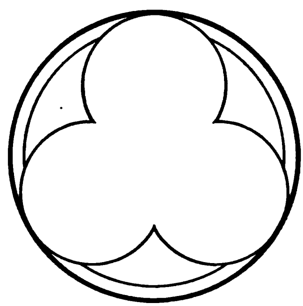
8. Draw three circles such that each passes through the centers of the other two.

9. Draw four circles such that each intersects the other three.

10. Classify triangles according to their angles; according to their sides.

11. With the aid of ruler and compasses draw the following figures:

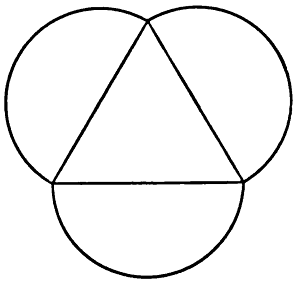


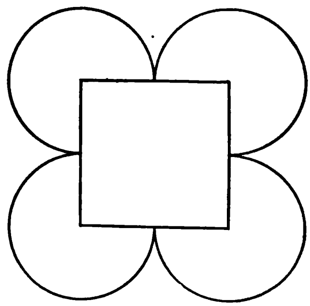


In the first figure the radii of all the arcs are equal, the centers being respectively the vertices of the square and the mid-points of the sides. The second figure is based on an equilateral triangle, the centers of the interior arcs being the mid-points of radii drawn to the vertices of the equilateral triangle.

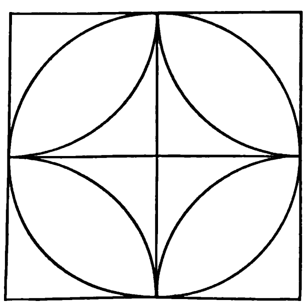
12. Lay off on a straight line *AB* ten successive centimeter divisions, and mark the points of division 0, 1, 2, 3, 4, $\cdots$. With point 2 as a center, and with a radius of 2 cm., draw a semicircle terminated by points 0 and 4. On the same side of *AB*, and with point 8 as a center, draw a semicircle like the first. With point 5 as a center, and on the same side of AB as before, draw a semicircle passing through the points 2 and 8. Then, using the points 1, 5, and 9 as successive centers, and with a radius of 1 cm., draw semicircles on the opposite side of AB. The completed figure should present the appearance of a closed curve.

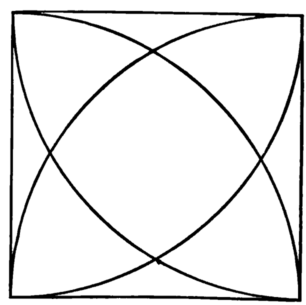
13. In the first of the two following designs locate by measurement the centers of the arcs surrounding the equilateral triangle. In the second design the vertices of the square are the centers of the arcs. Draw the figures.





14. With the aid of ruler and compasses draw the following figures:





In each of these figures the vertices of the square are used as centers for four of the arcs. In the second figure the radius equals one side of the square.

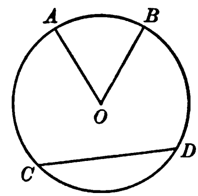
15. Complete the circles in the figures for Ex. 13.

CIRCLE AND ANGLE

PRELIMINARY DEFINITIONS

80. An angle formed by two radii is called a **central angle.**

81. A central angle is said to **intercept** the arc cut off by its sides.

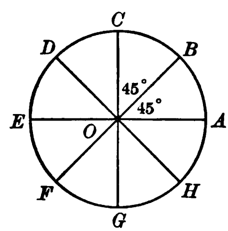


82. A chord is said to **subtend** the arc whose extremities it joins.

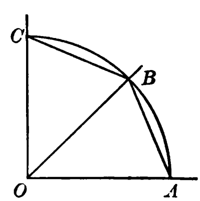
83. Two arcs are said to be **complements, supplements, conjugates,** of each other, according as their sum is a quadrant, a semicircle, a circle, respectively.

84. Of two unequal conjugate arcs of a circle, the less is called the **minor arc** and the greater the **major arc.** A minor arc is generally called simply an arc.

85. Relation of Central Angles, Arcs, and Chords. Let the figure suggest a wheel with eight spokes, the angles at the center being equal, each containing 45°. Copy the figure on a piece of tracing paper, the point corresponding to A being named A', etc. Revolve the duplicate figure around the center O. The circle will move on itself. When the point A' reaches the position B, point B' will fall on C, etc. Hence the arc AB is seen to coincide with arc BC. (Why?) From this we infer that



Equal central angles in a circle intercept equal arcs. Conversely, *equal arcs of a circle are intercepted by equal central angles.*



86. If in the previous figures we had drawn the chords subtending the arcs AB, BC, etc., we could have proved at the same time that the equality of the arcs involves also the equality of their subtending chords. For when the extremities of two equal arcs of a circle coincide, their chords also coincide. (Why ?) Hence

Equal chords of a circle subtend equal arcs ; and conversely, *equal arcs are subtended by equal chords.*

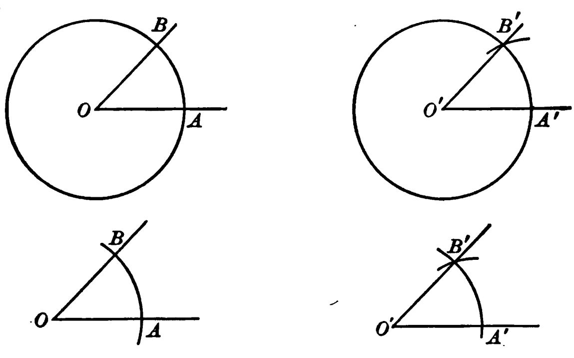
87. These relations of arcs, chords, and central angles evidently hold also for equal circles. They enable us to *construct* an angle equal to a given angle; to twice a given angle, etc.

88. Summary. *In the same circle or in equal circles:*

1. *Equal central angles intercept equal arcs.*
2. *Equal arcs are intercepted by equal central angles.*
3. *Equal chords subtend equal arcs.*
4. *Equal arcs are subtended by equal chords.*

Resulting Constructions

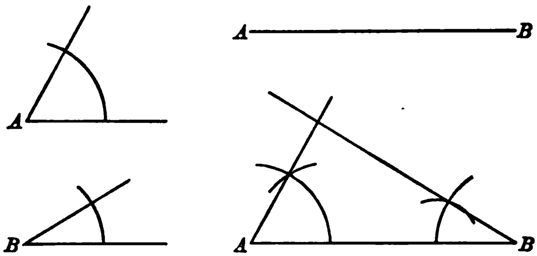
89. Construction IV. *To construct an angle equal to a given angle.*



Let $\angle AOB$ be the given angle. With center O and any radius OA, describe a circle. With O' as a center, and using a radius $O'A'$ equal to OA, describe a circle. Make chord $A'B'$ equal to chord AB by drawing an arc with center A' and radius AB. Then $\angle A'O'B' = \angle AOB$. (Why ?)

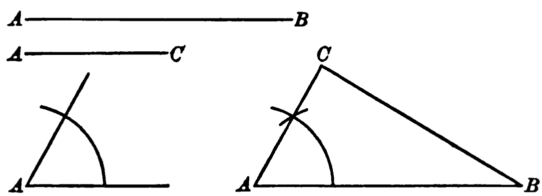
Is it necessary to draw the entire circle ?

90. Construction V. *To construct a triangle, given a side and the two adjoining angles.*

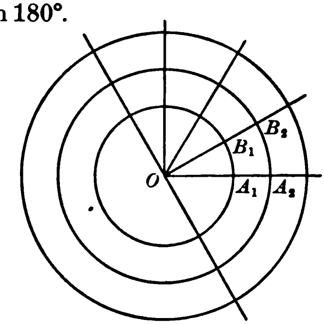


The diagram explains the construction. It is possible when the sum of $\angle A$ and $\angle B$ is less than 180°.

91. Construction VI. *To construct a triangle, given two sides and the included angle.*



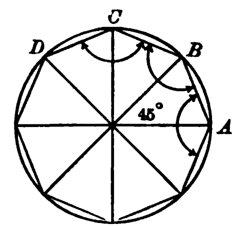
The diagram explains the construction. It is possible for all values of AB, AC, and $\angle A$ less than 180°.



92. Theory of the Protractor. Construct a number of equal angles having a common vertex O. With center O and arbitrary radii draw concentric circles. The result is seen in the figure. The equal angles intercept equal arcs on each of the circles. If, now, there had been 360 equal angles about O, that is, 360 angle-degrees, each circle would have been divided into 360 equal arcs. These 360 equal circle divisions are also called **degrees**. An arc-degree is $\frac{1}{360}$ of a circle. Thus if $\angle A_1OB_1$ contains 30 angle-degrees, arc A_1B_1 contains 30 arc-degrees. This explains why two arcs, e.g. A_1B_1 and A_2B_2, may contain the same number of arc-degrees without being equal. It also explains why circular protractors of different sizes can be used for drawing an angle of a given number of degrees. In general,

93. *The number of angle-degrees in the central angle is equal to the number of arc-degrees in the intercepted arc.* In other words,

94. *A central angle is measured by its intercepted arc.*



95. Regular Polygons. The chords of the arcs intercepted by equal central angles of a circle are equal. Also the base angles of the isosceles triangles thus formed can be proved equal by folding. Hence the doubles of these base angles are equal; for example, $\angle ABC = \angle BCD$. Whenever the series of equal chords forms a closed perimeter, a regular polygon results; for the sides and the angles of such a polygon are equal. This will always happen when the first central angle is contained an integral number of times in a round angle.

EXERCISES

Angle Constructions

(In the following exercises the protractor should not be used.)

1. At a given point in a line construct an angle equal to a given angle. Show that this may be done in four ways.

2. Construct an angle five times as large as a given acute angle.

3. Construct an angle equal to the difference of two given angles.

4. Construct the difference of an interior angle of a regular hexagon and an interior angle of a regular pentagon. Is the resulting angle contained exactly in a round angle?

5. Double a given arc of a circle.

6. Construct the difference of two arcs of a circle.

7. Construct an angle equal to the sum of the angles of a triangle. What do you observe?

8. Construct the sum of the angles of a quadrilateral.

9. Construct the sum of three consecutive angles of a regular hexagon.

Computations

10. How many degrees in a quadrant?

11. How many degrees in the arcs subtended by the sides of a regular polygon of 4 sides? 5 sides? 6 sides? 10 sides? 15 sides? 24 sides? n sides?

12. Through how many degrees does a point on the equator turn in 4 hr.? 6 hr.? 12 hr.? 18 hr.? 1 min.? 10 min.?

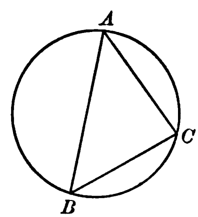
13. The length of the sun's equator is about 2,722,000 mi., while that of the earth's equator is about 25,000 mi. How many miles in a degree of each of these equators?

14. The sun's time of rotation is about 25 days. In what time does a sun spot near the equator pass through 72°? How many miles does it traverse during that time? What are the corresponding answers for a point on the earth's equator?

15. If the earth's orbit around the sun is regarded as a circle, how many degrees does the earth traverse in one month? in one day?

16. If the sun's apparent path across the sky each day is regarded as a semicircle, how many degrees does the sun traverse in 1 hr. on a day which is 12 hrs. long?

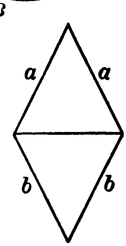
Review and Extension



17. Triangle ABC is said to be **inscribed** in the circle. Inscribe a triangle in a circle and then copy the figure.

18. How may a segment of given length be made a chord of a given circle?

19. Lay off the radius of a circle as a chord six times in succession. What do you observe? Can you give a reason for this?

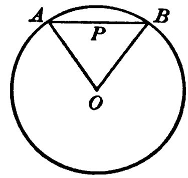


20. Construct two isosceles triangles on opposite sides of a common base. Join their vertices.

21. If $a = b$ in the above exercise, show that each triangle is but a repetition of the other. Under what circumstances can a circle be circumscribed about the entire figure?

22. Inscribe a quadrilateral in a circle. Construct the sum of two opposite angles of the figure.

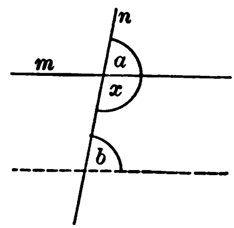
23. If the center O of a circle is joined to the extremities of a chord AB, prove that an isosceles triangle is formed.



24. If the isosceles triangle OAB in Ex. 23 revolves about O, prove that A and B move on the circle. What kind of figure does the mid-point P of AB describe?

25. In a semicircle inscribe a triangle, making the diameter one side of the triangle.

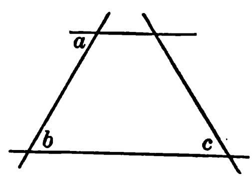
26. Prove, by folding, that any diameter of a circle bisects the circle.



27. Two lines, m and n, intersect, one of the angles formed being $\angle a$. At any point of n construct $\angle b = \angle a$. Which other angles in the figure are equal to $\angle a$ and $\angle b$? How is $\angle x$ related to $\angle b$?

28. Construct a quadrilateral (see accompanying figure) such that

$$\angle a + \angle b + \angle c = 2 \text{ rt. } \angle s.$$



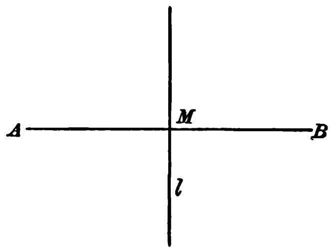
29. How many of the geometric ideas developed so far do you find illustrated concretely in the schoolroom?

30. Prepare a summary of the most important definitions, constructions, and propositions in Part I.

PART II

AXIAL SYMMETRY

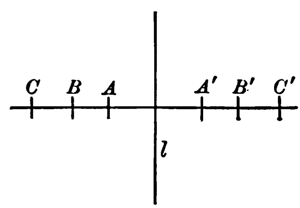
96. Preliminary Exercises. (1) Fold a sheet of paper, and before opening prick a hole through it. Then open. There will be two holes, one on each side of the crease. Name these A and B, and name the line of the crease, l. Draw the segment AB, and name the point where it crosses the crease, M. Show that (1) M is the mid-point of AB; (2) l is perpendicular to AB.



The line of folding is the perpendicular bisector of the segment AB.

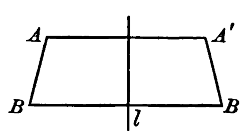
(2) In (1) point A is said to **correspond** to point B. This is sometimes indicated by the abbreviation $A \mid B$. Extend AB indefinitely. Locate three pairs of corresponding points.

In the figure, $A \mid A'$
$B \mid B'$
$C \mid C'$

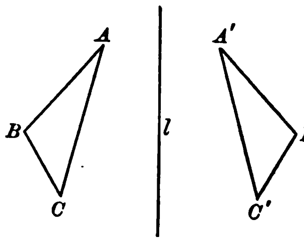


(Read "A corresponds to A'," etc.)

(3) Fold a piece of paper, and before opening prick two holes. Upon opening you will find two pairs of holes. Mark these A, A', B, B', as in (2). Connect A and A', B and B'. Then $AB \mid A'B'$ (read "AB corresponds to $A'B'$."). It is evident that to every point on AB there corresponds a point on $A'B'$, and that $AB = A'B'$. Also l is the perpendicular bisector of AA' and BB'.



(4) Repeat (3), pricking three holes. There will be three pairs of corresponding points. Join these as in (3). $ABC|A'B'C'$; that is, to every point of $\triangle ABC$ there corresponds a point of $\triangle A'B'C'$. $\triangle ABC$ may be made to coincide with $\triangle A'B'C'$. In the same manner we could obtain a polygon $ABCDE \cdots | A'B'C'D'E' \cdots$. The two figures so obtained could be made to coincide.



Summary

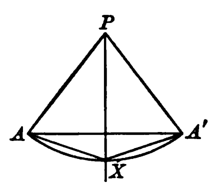
97. The corresponding points in the preceding figures are often called **symmetric points.** The line of folding is called the **axis of symmetry.**

98. Two points, A and A', are said to be **symmetric with respect to a line** l, called their axis of symmetry, if that line is the perpendicular bisector of the segment joining the two points. In general, two figures are symmetric with respect to a line, called their axis of symmetry, if to every point A of the first there corresponds a point A' of the second, such that the axis is the perpendicular bisector of the segment joining the two points.

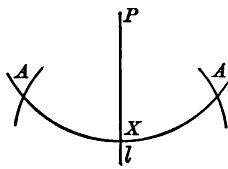
99. *Two symmetric (plane) figures can be made to coincide.*

Axial symmetry is of great significance, as is apparent in the symmetric structure of the human body, of animal organisms, of leaves, etc. Axial symmetry is used extensively in architecture, designing, pattern making, etc.

100. Construction of Symmetric Figures. It is evident that if P is *any* point in the axis, $PA = PA'$; that is, P is equidistant from A and A'. A circle of radius PA and center P will pass through A'. This circle must intersect the axis at some point X. Then $XA = XA'$. These relations enable us to construct a point A' symmetric to a given point A.



101. *To construct A' so that A' | A, when A is a given point and l is the given axis.* From any point P in the axis as a center, with a radius PA, draw an arc cutting the axis at X. From X as a center, with a radius XA, draw an arc intersecting the first arc at A'. A' is the symmetric point required. (Why?)



EXERCISES

1. Construct a segment $A'B'$ so that $A'B' \mid AB$.

2. Construct a triangle $A'B'C'$ so that $A'B'C' \mid ABC$.

3. Construct a quadrilateral $A'B'C'D'$ so that $A'B'C'D' \mid ABCD$.

4. Place a plane mirror so that it is perpendicular to a sheet of paper. Mark a point on the paper. Observe that the point and its image are symmetrically situated with respect to the base of the mirror. This is true of any figure drawn on the paper. Illustrate by drawing on your paper a square; a pentagon; a hexagon.

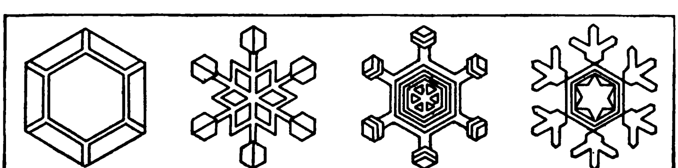
5. If you had accidentally closed your notebook before allowing the ink to dry, each letter would have made a symmetric impression. How is the same principle applied in printing?

6. Fold a sheet of paper and cut out any figure whatsoever, beginning at the crease and returning to it. A symmetric pattern will be obtained. How is the same principle applied in tailoring?

7. Study the capital letters of the alphabet. How many do *not* show axial symmetry?

8. What geometrical principle is illustrated by a snow crystal? (See figures below.)

9. The figures for the snow crystals show that more than one axis of symmetry may be present. How many are there in each case?



102. Conclusions. The following statements will now be readily understood.

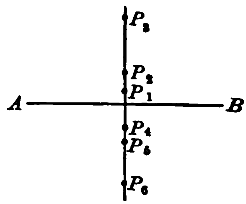
1. *Any point in the axis of symmetry of two points A and B is equidistant from them.*

2. *Any point equidistant from two given points lies in their axis of symmetry.*

3. *Hence the axis of symmetry of two given points contains all the points equidistant from them.*

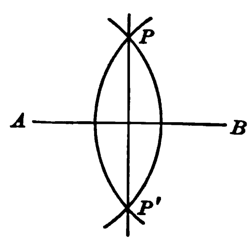
4. *Two points each equidistant from the extremities of a segment determine its perpendicular bisector.*

The equivalents of the four statements preceding will be considered fully in Book I.

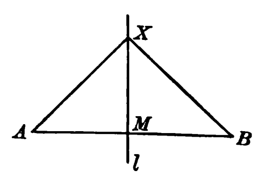


103. Construction of the Axis. In the figure, P_1, P_2, P_3, etc., are points equidistant from A and B. We have seen that they all lie in the perpendicular bisector of AB. But since two points determine a line, two of these equidistant points determine the perpendicular bisector of AB.

Hence, to construct the axis of symmetry of two points, A and B, construct *two* points P and P', each equidistant from A and B. The line passing through these points is the required axis.



104. The Isosceles Triangle. The properties of the axis of symmetry of two points are closely connected with the isosceles triangle. In the figure, l is the axis of A and B, X is *any* point of the axis, M is the mid-point of AB. The truth of the following statements is at once apparent.



1. *X is the vertex of an isosceles triangle whose base is AB.*

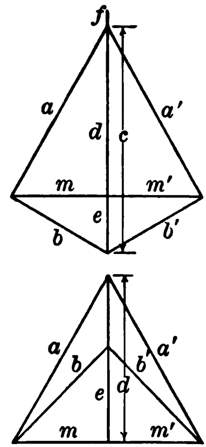
2. *Every isosceles triangle has an axis of symmetry, determined by its vertex and the mid-point of its base.*

3. *The base angles* ($\angle A$ *and* $\angle B$) *of an isosceles triangle are equal.*

Prove by folding.

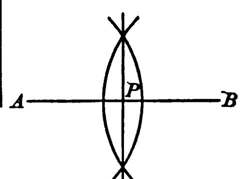
4. *The perpendicular bisector of the base of an isosceles triangle passes through the vertex and bisects the vertex angle.*

105. The Kite. If *two* points of the axis are joined to the symmetric points, we obtain two isosceles triangles having a common base. Their vertices may lie on the *same* side of the base or on *opposite* sides. The result in the latter case is a kite-shaped figure. Since these figures will serve as the foundation of the most fundamental constructions, we shall now give a table of their most important properties, using the accompanying diagrams.



Segments	Angles
$a = a'$	$\angle am = \angle a'm'$
$b = b'$	$\angle bm = \angle b'm'$
$m = m'$	$\therefore \angle ab = \angle a'b'$. Why?
	$\angle ad = \angle a'd$
	$\angle be = \angle b'e$

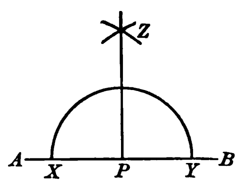
FUNDAMENTAL CONSTRUCTIONS



106. *To bisect a given segment.*

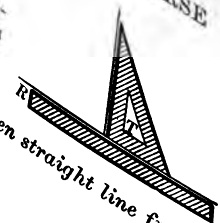
Construct the perpendicular bisector of AB.

107. *At a given point of a straight line to erect a perpendicular to that line.*



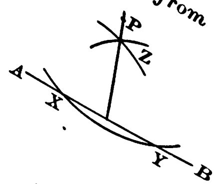
Take the given point P as a center, and with any convenient radius draw arcs cutting the line AB at X and Y. Then construct Z equidistant from X and Y. Join P and Z. PZ is the required perpendicular. (Why?)

108. By the previous construction it is possible to make a right triangle. Such a triangle may be made of cardboard or wood and used as a pattern for the construction of perpendiculars. This is shown in the figure. *R* is a ruler placed along the given line and *T* is the triangle.



109. *To drop a perpendicular to a given straight line from a given external point.*

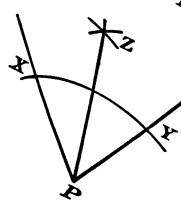
First Method. From *P* as a center, with a convenient radius, describe an arc cutting *AB* at *X* and *Y*. Then construct *Z* equidistant from *X* and *Y*. *PZ* is the required perpendicular. (Why?)



Second Method. Use a right triangle.

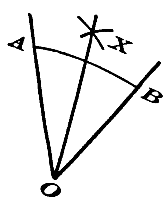
110. *To bisect a given angle.*

From *P* as a center, with a convenient radius, describe an arc cutting the sides of the angle at *X* and *Y*. Construct *Z* equidistant from *X* and *Y*. *PZ* bisects the given angle. (Why?)



111. *To bisect a given arc.*

If *O* is the center of the arc *AB*, bisect the ∠*AOB*. *OX* bisects the arc. (Why?)



Note. The five constructions of §§ 106–111 will be considered again in Book I. At this point they are intended to give practice and to develop greater skill in using the geometric instruments, viz., the straightedge and the compasses.

EXERCISES

1. Show by repeated bisection that a segment can be divided into 2, 4, 8, 16, 2^n equal parts. Explain.

2. An angle can be divided into 2, 4, 8, 16, 2^n equal parts.

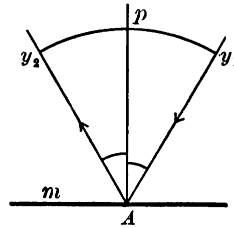
3. Construct angles of 90°, 45°, $22\frac{1}{2}$°, 135°, without using a protractor.

Construct a compass card without using a protractor.

5. Assuming that the sum of the angles of a triangle equals 180°, show by symmetry that each angle of an equilateral triangle contains 60°.

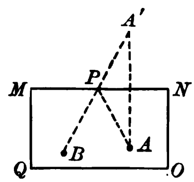
6. Construct an angle of 60°.

7. Construct angles of 30°, 15°, 75°, 105°. (75° = 60° + 15°.)



8. If a ray of light y_1 strikes a plane mirror at A, it is reflected in such a manner that $\angle py_1 = \angle py_2$; that is, the perpendicular p to m at A bisects $\angle y_1y_2$. Construct the figure.

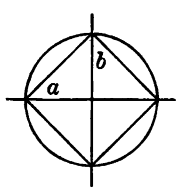
9. A ray of light is reflected by a mirror m_1 into a mirror m_2. Construct the path of the ray.

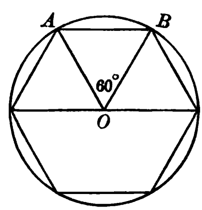


10. The figure $MNOQ$ represents a billiard table. Required to strike the ball A in such a manner that it will rebound from MN and hit the ball B. Construct the path of A.

Solution. Construct $A'|A$. Draw $A'B$, cutting MN at P. APB is the path of the ball A.

11. We can now construct a number of regular polygons without using a protractor. The first figure shows how a square may be constructed, by making a perpendicular to b, etc. The second figure shows that if $\angle O = 60°$, $\triangle AOB$ is equilateral (see Ex. 5). Hence AB equals OA, the radius of the circle. Copy each of the figures.



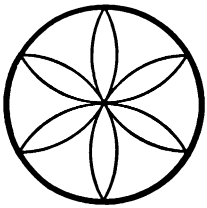


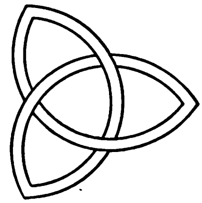
12. Construct polygons of 4, 8, 16 sides.

13. Construct polygons of 6, 12, 24 sides.

14. How many axes of symmetry does a square have? a regular pentagon? a regular hexagon? a circle?

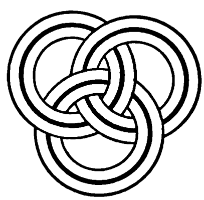
15. With the aid of ruler and compasses draw the following figures:



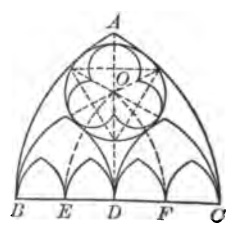


The first two figures are based on the fact that the side of a regular hexagon is equal to the radius of the circumscribed circle. In the third figure the mid-points of the sides of an equilateral triangle are the centers of the arcs.

16. With the aid of ruler and compasses draw the following figures:

In the first figure the vertices of the six-pointed star, including the points of intersection of the sides of the two equilateral triangles, are the centers of the arcs. The second figure arises from completing the arcs of the third figure of the preceding exercise.

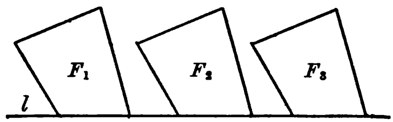


17. The figure shows the design of a Gothic window. The outer arcs AC and AB are constructed with B and C as centers. The small arcs have as centers the points B, C, D, E, F. The center O is obtained by drawing arcs with B and C as centers and with a radius equal to BF. Draw the figure.

(In order to lessen the difficulty of this exercise, the figure should be drawn on a large scale.)

CONGRUENCE

112. Suppose that a paper-cutting machine is made to cut out a pattern from a stack of paper. The pattern will then appear as many times as there are sheets of paper. Imagine all these figures placed along a line l.



The figures *agree* in all respects, since they are but *repetitions* of one figure. This could at any time be verified by sliding the figures back on the line l to their original position F_1. Each is then seen to be a *duplicate* of F_1. Such figures are called congruent (from the Latin *congruere*, "to agree").

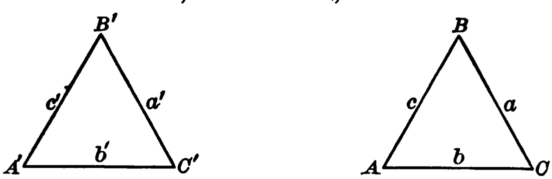
113. Two figures which may be made to coincide in all their parts are said to be **congruent**. We shall use the symbol $\equiv$ for the word "congruent." $F_1 \equiv F_2$ means that F_1 is congruent to F_2. From this definition it follows that *figures congruent to the same figure are congruent to each other.*

114. Homologous Sides and Angles. From the definition it follows that if two figures are congruent, each side and angle of one has a corresponding equal side and angle in the other.

Thus if $\qquad \triangle ABC \equiv \triangle A'B'C',$

then $\qquad a = a', \qquad b = b', \qquad c = c',$

and $\qquad \angle A = \angle A', \quad \angle B = \angle B', \quad \angle C = \angle C'.$

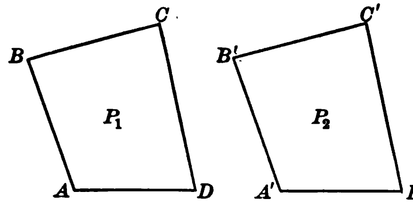


The sides and angles of a figure are called its **principal elements** or **parts**. Any two corresponding parts, such as a and a', are said to be **homologous**.

115. Construction. *To construct a polygon congruent to a given polygon.*

If P_1 is the given polygon, construct

$A'B' = AB,$
$\angle A' = \angle A,$
$\angle B' = \angle B,$
$A'D' = AD,$
and $B'C' = BC.$



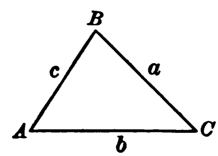
It will then be found that the last side ($C'D'$) and the two remaining angles ($\angle C'$ and $\angle D'$) need not be constructed, as they are already determined or fixed by the preceding elements of the figure. The new polygon, if it were placed on the given polygon, would coincide with it exactly. Hence $P_2 \equiv P_1$.

It is readily seen that even if P_1 had contained a larger number of sides, we should have arrived at the same conclusion; namely, that the last three elements of a polygon are determined by the preceding ones. Now a polygon of n sides has n angles, or $2n$ parts. Hence two polygons are congruent if they agree in $2n - 3$ consecutive elements.

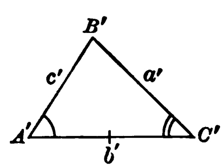
116. *To construct a triangle congruent to a given triangle.*

Let ABC be the given triangle, in which

$a = 6$ cm.
$b = 7$ cm.
$c = 5$ cm.



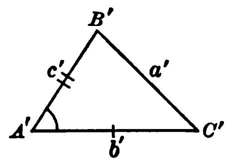
First Method. Draw $b' = b, \angle A' = \angle A, \angle C' = \angle C$. Continue the sides of $\angle A'$ and $\angle C'$. They will intersect at a point B'.



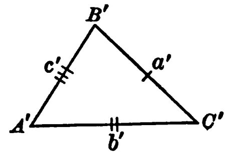
The result is the triangle $A'B'C'$. It can be shown by **superposition** (that is, by *placing* one triangle *on* the other) that the two triangles are congruent.

Note. $\triangle A'B'C'$ may be constructed congruent to $\triangle ABC$ with the order of the parts the *same* or the *reverse* of that in $\triangle ABC$ (compare § 99).

SECOND METHOD. Draw $b' = b$, $\angle A' = \angle A$, and $c' = c$. Connect B', the extremity of c', with C', the extremity of b'. A definite triangle $A'B'C'$ results. The congruence of the two triangles can be tested by superposition.



THIRD METHOD. Draw $b' = b$, and with centers A' and C' and radii c and a respectively, describe arcs. These intersect at B'. It will be shown later (§ 142) that in this case also



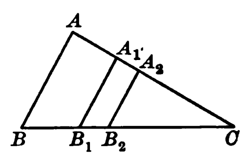
$$\triangle A'B'C' \equiv \triangle ABC.$$

117. These three constructions suggest the following very important laws of congruence of triangles:

First Law of Congruence. *If two triangles have a side and the two adjoining angles of one equal respectively to a side and the two adjoining angles of the other, the triangles are congruent.* (a. s. a.)

Second Law of Congruence. *If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent.* (s. a. s.)

Third Law of Congruence. *If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are congruent.* (s. s. s.)



REMARK 1. The congruence of two triangles cannot be made to depend upon angles only. The diagram shows that triangles may be mutually equiangular without being congruent.

REMARK 2. The frequent use of these laws of congruence makes abbreviation desirable. Thus the first will be represented by the abbreviation "a. s. a.," that is, the law of one side and the two adjoining angles; the second, by "s. a. s."; the third, by "s. s. s."

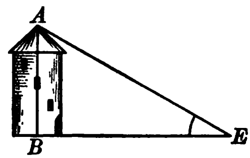
REMARK 3. Formal proof of these three laws is given among the propositions of Book I.

EXERCISES

1. Construct a $\triangle ABC$ such that $a = 3$ in., $\angle B = 45°$, $\angle C = 30°$. Repeat your construction and test the congruence of the triangles by superposition (a. s. a.).

2. Construct $\triangle ABC$ so that $a = 3$ in., $B = 22\frac{1}{2}°$, $c = 5$ in. Repeat your construction, and by superposition show that the two triangles coincide (s. a. s.).

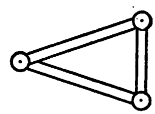
3. Construct a $\triangle ABC$ such that $a = 4$ cm., $b = 5$ cm., $c = 6$ cm. Repeat the construction and test with tracing paper whether or not the two triangles are congruent (s. s. s.).



4. Let AB represent a tower 100 ft. high. Ascertain by measurement from a drawing to scale how large an angle it subtends at the eye E of an observer if $BE = 200$ ft.?

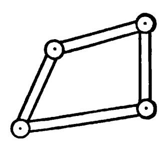
5. Construct a polygon congruent to a given polygon of six sides by means of s. s. s.

Suggestion. Draw diagonals and construct congruent triangles in their proper positions.



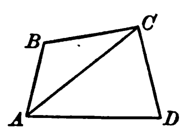
6. Two triangles, in order to be congruent, must have *three* parts respectively equal, and at least *one* side must be used. Explain.

7. What is meant by the statement, A triangular frame is rigid? Of what practical value is this fact in the construction of houses, bridges, truss-work?



8. Show that a four-sided frame becomes rigid if a diagonal crosspiece is introduced. How does a carpenter use this fact in making a gate?

9. The three laws of congruence may be replaced by the statement: A triangle is *determined* by (1) one side and two angles; (2) two sides and the included angle; (3) three sides. Explain.

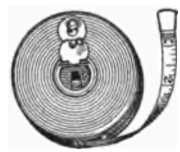


10. Construct a quadrilateral $ABCD$, (1) if $AB = 2$ in., $BC = 1$ in., $CD = \frac{3}{4}$ in., $DA = 1\frac{1}{2}$ in., $AC = 2$ in.; (2) if $AB = 5$ cm., $BC = 6$ cm., $CD = 4\frac{1}{2}$ cm., $DA = 5.6$ cm., $AC = 8$ cm.

SURVEYING

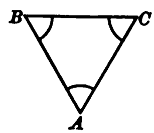
118. Surveying has for its purpose the determination, by processes of measurement, of the relative positions of points and lines upon the surface of the earth. A careful record must be kept of all measurements taken, in order that a picture or "plat" may be made of the lines or areas included in the survey.

119. Lines are measured with chains, tapes, or rods.



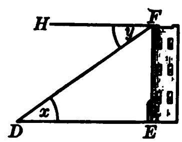
120. The angles to be measured directly are either **horizontal** or **vertical**.

121. Thus if A, B, C, are three points situated in the same horizontal plane (for example, on the surface of a floor), the three angles A, B, C of the triangle ABC are horizontal angles.



Again, if EF (see figure below) represents a tower standing on a horizontal plane DE, the angle FDE, or $\angle x$, is vertical.

122. Angles in the vertical plane are either **angles of elevation** or **angles of depression**.

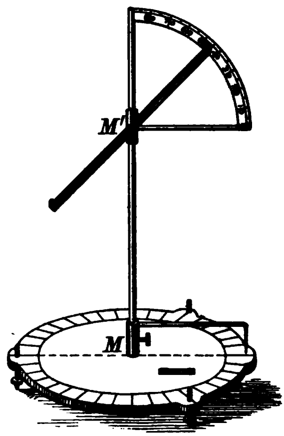


A person standing at D must look upward in order to see the point F, while a person standing at F must look downward in order to see the point D. Hence, $\angle x$ is called an angle of elevation, while $\angle y$ is an angle of depression.

123. The instrument commonly used by surveyors for measuring horizontal and vertical angles is called a **transit**. It consists essentially of a telescope free to move in a vertical circle and attached to a pointer which may be moved over a graduated horizontal circle. The circular plate of the transit can be made level by suitable attachments.

Let the center of the plate be denoted by O. Then if two points A and B are in the same plane with O, the telescope is pointed first at A and then at B, and the difference between the two readings on the graduated circle is determined.

This difference is the value of the horizontal angle AOB. Vertical angles are measured similarly by means of the vertical circle.

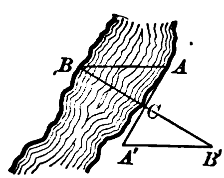


124. A simple combination instrument for measuring either horizontal or vertical angles is shown in the figure. This instrument will serve as an inexpensive substitute for a transit. The vertical rod MM' is free to revolve in the socket at M, carrying a horizontal pointer which indicates readings on the horizontal circle divided into degrees. These divisions must be marked. The pointer at M' is provided with sights and is free to move in a vertical circle around M'. By sighting along this pointer, vertical angles may be measured.

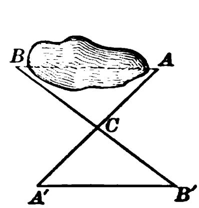
By the aid of some such instrument as this, and of a 50-ft. tape, a few rods and some stakes, distances and angles as seen from some given point in the school yard should be measured.

EXERCISES

1. In the diagram A and B are points separated by a river. How may the length of AB be determined by the first law of congruence? (a. s. a.) (A, B, C, represent trees or vertical rods. Measure AC and $\angle A$. Make $A'C = AC$, and $\angle CA'B' = \angle A$. $B'CB$ is a straight line. Measure $A'B'$. Then $A'B' = AB$.)



2. In the diagram AB represents the distance across a pond. The length of AB is to be determined by the second law (s. a. s.). A vertical rod is placed at C. Measure AC and BC. Make $A'C = AC$, and $B'C = BC$. ACA' and BCB' are straight lines. Then $A'B' = AB$. If $AC = 300$ ft., $BC = 200$ ft., $\angle C = 55°$, determine AB by measurement.



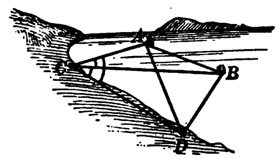
3. Explain what measurements are necessary in order to determine the distance of an anchored ship from the shore.

Suggestion. Use the method of a. s. a.

4. What measurements are necessary in order to determine the length of a rectangular building whose corners only are accessible?

Suggestion. Use the method of s. a. s.

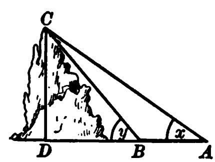
5. It is required to find the distance between two ships A and B anchored off the shore.



Solution. Measure a convenient length CD along the shore. With the instrument at C, and then at D, measure two angles at each point. For example, at C, measure any two of the angles ACB, ACD, BCD. Explain how AB is determined by these five measurements.

6. Find the height of a flagpole by measuring a convenient distance from its foot, and then measuring the angle of elevation of the top of the pole. If the distance is 150 ft. and the angle is 28°, how high is the pole? Measure by drawing to scale.

7. Determine the height of your high-school building by a drawing made to scale.



8. The height of a mountain CD may be found by determining its angle of elevation from two different points, A and B, situated in the same vertical plane with C. Then, if AB is known, CD can be constructed. Explain.

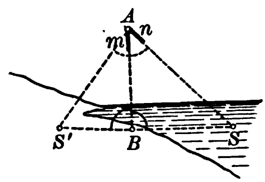
9. Explain how it is possible to find from the ground the height of a kite, a balloon, or a cloud.

10. Two persons 600 ft. apart, situated on *opposite* sides of a balloon as seen from the ground, observe its angles of elevation to be 44° and 55°. What is the height of the balloon? Draw to scale.

11. By what methods can the distance between the tops of two church steeples be determined?

12. How may the distance between two mountain tops be determined?

13. Thales of Miletus (600 B.C.) is said to have invented a way of finding the distance of a ship from the shore. His method is supposed to have been based on the law of a. s. a., and may have been as follows: Two rods, m and n, are hinged together at A. One arm, m, is held vertically, while the other, n, is pointed at the ship S. Then the instrument is revolved about m as an axis until n points at some familiar object S' on the shore. Then $\triangle ABS \equiv \triangle ABS'$, and $BS' = BS$. Explain.



14. The **bearing** of a place A from another place B is the angle of deviation of the line BA from the true north-and-south line through B. A is 5 miles west of B, and C is 9 miles south of A. Find, by measurement with the protractor, the bearing of C from B; of B from C. (Use a scale of two miles to the inch.)

15. A is 11 miles from B, and its bearing from B is N. 17° E. C is 9 miles from A, bearing S. 43° E. Find, by construction and measurement with ruler and protractor, the bearing of C from B; of B from C; and the distance BC.

Suggestion. Draw the north-and-south lines through the given points. These will be at right angles to an east-and-west base line at the bottom of the figure.

16. A surveyor maps a field by taking the successive bearings and distances from one corner to the next. Draw a map from the following survey notes, using a scale of 200 ft. to the inch, and determine by measurement the missing data.

Station occupied	Point sighted at	Bearing	Distance
A	B	N. 34° E.	450 ft.
B	C	N. 76° E.	200 ft.
C	D	S. 12° E.	600 ft.
D	E	S. 62° W.	225 ft.
E	A	?	?

17. The bearing of Long Branch, New Jersey, from Newark is S. 18° E. and its distance is 30 miles. The bearing of Trenton from Long Branch is S. 84° W. and its distance is 41 miles. Find the bearing and the distance of Newark from Trenton. (Use any convenient scale.)

BOOK I

RECTILINEAR FIGURES

A System of Propositions

125. In the preceding pages many simple geometric principles were presented and illustrated. There remain, however, many other facts of geometry which are not so apparent, or which, even though they seem quite easy of comprehension, are not easily proved.

The aim of **demonstrative geometry** is to point out methods of discovery and of proof, as well as a logical arrangement, of those geometric principles which are most important in the development of the subject, and which have the widest application in other fields of investigation. Such principles are called **propositions,** and, except for those which are taken for granted at the outset, are established by means of **formal demonstrations.**

126. A complete demonstration demands a *reason* for every statement which is included in it. It is obvious that such a justification of the particular steps of a proof is greatly facilitated by arranging the propositions in a *definite progressive order.*

127. The most fundamental propositions, which are first considered, are those which are most often in demand as the work progresses; much as the multiplication table is needed in the study of arithmetic. Other propositions are shown to depend more or less directly upon these, and thus gradually arises a system of propositions, each of which depends upon some or all of those preceding, in the order of consideration. For this reason the study of geometry has been compared to the building of a house or other structure; each stage of the work is supported by what has already been completed.

ARRANGEMENT OF A DEMONSTRATION

128. The formal demonstration of a theorem in geometry involves three steps:

1. The statement of what is given (the **hypothesis**).
2. The statement of what is to be proved (the **conclusion**).
3. The **proof**, consisting of several steps, each based upon the hypothesis, or upon the authority of definitions, axioms, preliminary theorems, or preceding propositions.

These three steps are indicated by the words

Given, **To prove,** **Proof.**

129. In order to avoid a mechanical repetition of the text, and to gain the ability to reason independently in the field of geometry, it is advisable for the student to bear in mind the following

RULES

1. Read the proposition carefully, noting the meaning of each term employed.

2. Draw a figure which represents the conditions demanded by the proposition, and no other special conditions.

3. State the hypothesis and the conclusion as applied to that particular figure.

4. Recall the geometric processes which generally lead to this conclusion, and examine the hypothesis and its immediate consequences in order to discover whether such processes may be applied in the particular instance.

5. Try to formulate a proof before examining the one given in the text.

6. Write out the proof, arranging the steps in order and numbering the main divisions. The final step should agree with the conclusion. After each step give the authority which supports it. Proper authorities are the hypothesis and any definitions, axioms, preliminary theorems, and propositions previously established. Use freely such symbols and abbreviations as are suggested below.

7. At a later period in the development of the work it is often sufficient, and even preferable, to write merely a brief summary of the demonstration.

8. As a review exercise, practice reciting the proof from a mental figure only.

SYMBOLS AND ABBREVIATIONS

$+$	plus, or added to.	adj.	adjacent.
$-$	minus, or diminished by.	alt.	alternate.
$=$	equal, or is equal to.	ax.	axiom.
$>$	is greater than.	circum.	circumference.
$<$	is less than.	comp.	complement, or complementary.
$\equiv$	is congruent to.	cons.	construction.
$\therefore$	therefore, or hence.	cor.	corollary.
$\perp$	perpendicular, or is perpendicular to.	corr.	corresponding.
		def.	definition.
$\perp$s	perpendiculars.	ex.	exercise.
$\parallel$	parallel, or is parallel to.	ext.	exterior.
$\parallel$s	parallels.	hom.	homologous.
$\backsim$	is similar to, or similar.	hyp.	hypothesis.
$\angle$	angle.	int.	interior.
$\angle$s	angles.	isos.	isosceles.
$\triangle$	triangle.	prop.	proposition.
$\triangle$s	triangles.	rect.	rectangle.
▱	parallelogram.	rt.	right.
▱s	parallelograms.	sq.	square.
$\odot$	circle.	st.	straight.
$\odot$s	circles.	supp.	supplement, or supplementary.

a. s. a., having a side and the two adjoining angles of one equal respectively to a side and the two adjoining angles of the other.

s. a. s., having two sides and the included angle of one equal respectively to two sides and the included angle of the other.

s. s. s., having three sides of one equal respectively to three sides of the other.

rt. $\triangle$, h. l., being right triangles and having the hypotenuse and a leg of one equal respectively to the hypotenuse and a leg of the other.

rt. $\triangle$, h. a., being right triangles and having the hypotenuse and an adjoining angle of one equal respectively to the hypotenuse and an adjoining angle of the other.

AXIOMS. PRELIMINARY PROPOSITIONS

130. All demonstration finally depends on taking some truths for granted. Such truths are sometimes called *axioms.* They are assumed to be so clear in themselves that every one will accept them without proof.

Geometry is no exception to this general rule. It is, however, essential in elementary geometry that the facts taken for granted and without demonstration should be of a *simple* nature, so simple, in fact, as to be obvious if merely plainly stated. The geometrical facts which are assumed in Book I will now be tabulated under the heading Preliminary Assumptions and Propositions, and it will be observed that we are thus giving a recapitulation of some of the main results of the informal discussion of the Preliminary Course.

131. Preliminary Assumptions and Propositions.

1. *Through a given point an indefinite number of straight lines may be drawn* (§ 11).

2. *Two straight lines can intersect in but one point* (§ 11).

3. *One and only one straight line can be drawn through two given points* (§ 11).

4. *Two straight lines that have two points in common coincide throughout and form but one line* (§ 11).

5. *Two straight lines do not inclose a space* (§ 11).

6. *Two line-segments whose extremities can be made to coincide must coincide throughout* (§ 13).

7. *Two line-segments a and b must be in one of three relations to each other, namely,* $a > b$, $a = b$, *or* $a < b$ (§ 13).

8. *Line-segments may be added together. Of two unequal line-segments the smaller may be subtracted from the larger. Line-segments may be multiplied by a given number* (§ 16).

9. *All round angles are equal* (§ 22).

10. *All straight angles are equal* (§ 24).

11. *All right angles are equal* (§ 28).

12. *Two angles A and B must be in one of three relations to each other:* $\angle A > \angle B$, $\angle A = \angle B$, *or* $\angle A < \angle B$ (§ 32).

13. *At a given point in a given line only one perpendicular can be drawn to that line (in the same plane)* (§ 41).

14. *The complements of the same angle, or of equal angles, are equal* (§ 41).

15. *The supplements of the same angle, or of equal angles, are equal* (§ 41).

16. *Vertical angles are equal* (§ 41).

17. *If two adjacent angles have their exterior sides in a straight line, they are supplementary* (§ 41).

18. *If two adjacent angles are supplementary, their exterior sides lie in a straight line* (§ 41).

19. *Radii of the same circle are equal* (§ 60).

20. *At a given point in a given line a line may be drawn making with the given line an angle equal to a given angle* (§ 89).

21. *Figures congruent to the same figure are congruent to each other* (§ 113).

132. General Axioms. Further fundamental truths involving *magnitude* are as follows:

1. *Magnitudes which are equal to the same magnitude, or to equal magnitudes, are equal to each other.*

In other words, *A magnitude may be substituted for its equal.*

2. *If equals are added to equals, the sums are equal.*

3. *If equals are subtracted from equals, the remainders are equal.*

4. *If equals are multiplied by equals, the products are equal.*

5. *If equals are divided by equals, the quotients are equal.*

The divisor must not be zero.

6. *Like powers or like positive roots of equals are equal.*

7. *The whole of any magnitude is equal to the sum of all its parts.*

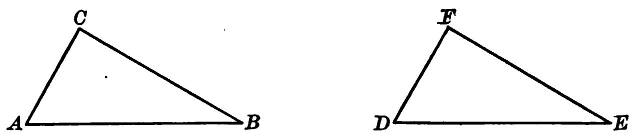
8. *The whole of any magnitude is greater than any part of it.*

CONGRUENCE

Proposition I. Theorem

FIRST TRIANGLE CONGRUENCE

133. *If two triangles have a side and the two adjoining angles of one equal respectively to a side and the two adjoining angles of the other, the triangles are congruent.* (a. s. a.)



Given the triangles *ABC* and *DEF*, in which *AB* equals *DE*, angle *A* equals angle *D*, and angle *B* equals angle *E*.

***To prove** that* $\triangle ABC \equiv \triangle DEF$.

Proof. 1. Place $\triangle ABC$ on $\triangle DEF$ so that AB coincides with DE, and C and F fall on the same side of DE.

2. Then AC will fall along DF,

($\angle A = \angle D$, by hyp.)

and BC will fall along EF.

($\angle B = \angle E$, by hyp.)

3. $\therefore C$ will fall upon F. § 11, (2)

(Two straight lines can intersect in but one point.)

4. $\therefore \triangle ABC$ coincides with $\triangle DEF$,

i.e. $\triangle ABC \equiv \triangle DEF$.

(Definition of congruence, § 113.)

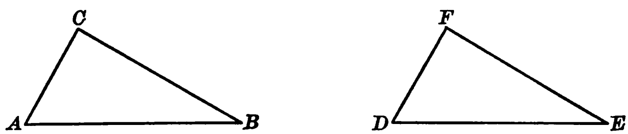
Remark. The method of proof adopted is **superposition** (§ 116), that is, "placing on"; namely, the triangle ABC was placed on the triangle DEF. The so-called **axiom of superposition** has been tacitly assumed. This axiom may be stated thus:

Any figure may be moved about in space without changing either its size or its shape.

PROPOSITION II. THEOREM

SECOND TRIANGLE CONGRUENCE

134. *If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent.* (s. a. s.)



Given the triangles *ABC* and *DEF*, in which *AB* equals *DE*, *AC* equals *DF*, and angle *A* equals angle *D*.

***To prove** that* $\triangle ABC \equiv \triangle DEF$.

Proof. 1. Place $\triangle ABC$ on $\triangle DEF$ so that AB coincides with DE, and C and F fall on the same side of DE.

2. Then AC will fall along DF.

($\angle A = \angle D$, by hyp.)

3. Also the point C will fall on the point F.

($AC = DF$, by hyp.)

4. $\therefore CB$ will coincide with FE. § 13, (4)

(Their extremities being the same points.)

5. $\therefore \triangle ABC$ coincides with $\triangle DEF$.

$\therefore \triangle ABC \equiv \triangle DEF$.

(Definition of congruence.)

135. Homologous parts (sides or angles) of congruent triangles are those parts which are opposite parts known to be equal. Thus, in the above figures, BC is homologous to EF, $\angle B$ to $\angle E$, and $\angle C$ to $\angle F$. Obviously

Homologous parts of congruent triangles are equal.

136. The congruence of two triangles affords a fundamental method of establishing the equality of two given lines or angles, as outlined in the following scheme:

In order to prove two $\left(\begin{matrix}\textbf{lines}\\ \textbf{angles}\end{matrix}\right)$ **equal:**

I. Show that they are homologous $\left(\begin{matrix}\textbf{sides}\\ \textbf{angles}\end{matrix}\right)$ **of congruent** $\triangle$**.**

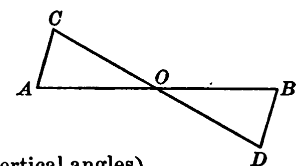
In order to prove two triangles congruent:

1. Show that they have the relation a. s. a.
2. Show that they have the relation s. a. s.

Note. It will also be helpful to recall, at this point, the cases of equal angles treated previously (e.g. right angles, vertical angles, etc.), as well as the equalities resulting from a use of the axioms of equality (§ 132).

EXERCISES

1. Given, in the annexed figure, that AB and CD bisect each other at O.



To prove that $AC = BD$.

Proof. 1. $\triangle AOC \equiv \triangle BOD$. s.a.s.

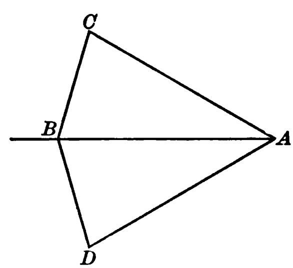
For $OA = OB$, Hyp.

$OC = OD$, Hyp.

and $\angle AOC = \angle BOD$ (being vertical angles).

2. $\therefore AC = BD$ (being homologous sides of congruent $\triangle$).

State this exercise in the form of a proposition.

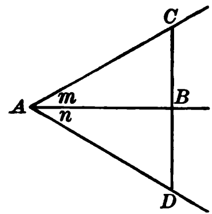


2. Given, in the figure for the preceding exercise, that O is the mid-point of CD, and that $\angle C = \angle D$. Prove that AB, cutting CD in O, is bisected at O.

3. Given, in the annexed figure, that the line AB bisects the angles CAD and CBD. Prove that $\angle C = \angle D$.

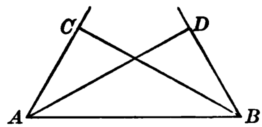
4. Given, in the figure for the preceding exercise, that $\angle CAD$ is bisected by AB, and that $AC = AD$. Prove that $BC = BD$.

5. Given, in the annexed figure, that $\angle m = \angle n$, $AC = AD$, and that the line CD intersects AB at B. Prove that $BC = BD$. State as a general theorem.



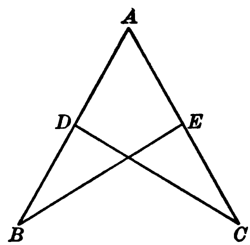
What angles also are proved equal? What kind of angle, therefore, is $\angle ABC$? What is the position of AB relative to CD?

6. Given, in the annexed figure, that $AC = BD$, and $\angle CAB = \angle DBA$. Prove that $BC = AD$. What other parts are equal?



7. Given, in the figure for Ex. 6, that $\angle CAB = \angle DBA$, and $\angle CAD = \angle DBC$. Prove that $BC = AD$.

8. Given, in the annexed figure, that $AB = AC$, and $AD = AE$. Prove that $BE = CD$. What other parts are equal?



9. Given, in the figure for Ex. 8, that $AB = AC$, and $\angle B = \angle C$. Prove that $\angle BDC = \angle CEB$. What other parts are equal?

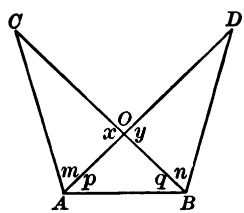
137. In order to obtain a sufficient number of equal parts of two triangles to establish either of the relations a. s. a. or s. a. s., it is sometimes necessary first to prove two other triangles congruent.

EXAMPLE

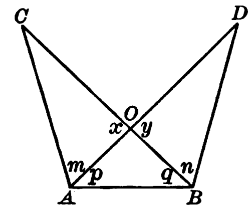
Given, in the annexed figure, that

$AC = BD$, and $\angle CAB = \angle DBA$.

To prove that $CO = DO$.



Analysis. CO and OD are parts of $\triangle COA$ and DOB respectively. In these triangles $AC = BD$ (hyp.), and $\angle x = \angle y$ (vert. $\angle$s), but these equalities are insufficient to establish the congruence of the triangles by either of the methods given. It is therefore necessary to prove two other triangles congruent in order to obtain a larger number of equal parts. Triangles ABC and ABD are chosen, since they are readily proved to be congruent. Hence the following:



Proof. 1. $\triangle ABC \equiv \triangle ABD$. s. a. s.

For $AC = BD$, Hyp.

AB is common,

and $\angle CAB = \angle DBA$. Hyp.

2. $\therefore \angle C = \angle D$, and $\angle p = \angle q$. Why?

3. Then $\angle CAB - \angle p = \angle DBA - \angle q$. Ax. 3

That is, $\angle m = \angle n$.

4. $\therefore \triangle COA \equiv \triangle DOB$. a. s. a.

5. $\therefore CO = DO$. Why?

EXERCISES

1. Given, in Fig. 1, that $\angle m = \angle n$, and $\angle x = \angle y$. Prove that $CB = BD$.

2. Given, in Fig. 2, that $AB = AC$, and $AD = AE$. Prove that $\angle ABC = \angle ACB$.

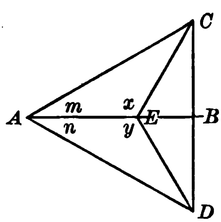


FIG. 1

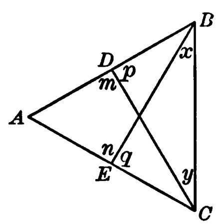


FIG. 2

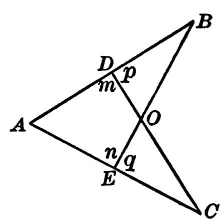


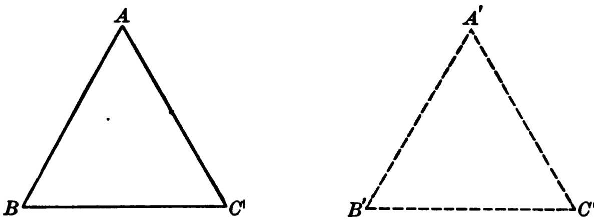
FIG. 3

3. Given, in Fig. 2, that $\angle DBC = \angle ECB$, and $\angle x = \angle y$. Prove that $AB = AC$.

4. Given, in Fig. 3, that $AB = AC$, and $AD = AE$. Prove that $BO = CO$.

Proposition III. Theorem

138. *If two sides of a triangle are equal, the angles opposite these sides are equal.*



Given the triangle *ABC*, in which *AB* equals *AC*.

***To prove* that** $\angle B = \angle C$.

Proof. 1. Construct the $\triangle A'B'C'$ congruent to $\triangle ABC$, by making $A'B' = AB$, $A'C' = AC$, and $\angle A' = \angle A$. s. a. s.

2. Then $\angle B'$ may be made to coincide with $\angle B$.

(Homologous angles of congruent $\triangle$s.)

3. But $\triangle A'B'C'$ may be made to coincide with $\triangle ABC$, even when turned over and placed so that B' falls on C and C' falls on B. s. a. s.

4. Then $\angle B'$ will coincide with $\angle C$. Why?

5. $\therefore \angle B = \angle C$. Ax. 1

(Each being equal to $\angle B'$.)

Remark. Proposition III may also be stated thus: *In an isosceles triangle the angles opposite the equal sides are equal.*

139. Corollary.* *An equilateral triangle is also equiangular.*

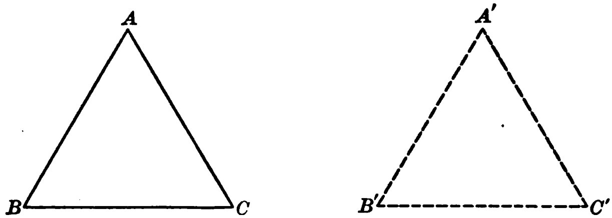
For any side may be regarded as the base of an isosceles triangle.

Remark. Note that Proposition III may be proved by applying the method of Ex. 2, p. 78. This is similar to the method of Euclid, Book I, Proposition V (the *pons asinorum*).

* A **corollary** is a truth readily deduced from a statement immediately preceding.

Proposition IV. Theorem

140. *If two angles of a triangle are equal, the sides opposite those angles are equal.*



Given the triangle *ABC*, in which angle *B* equals angle *C*.

***To prove** that* $AB = AC$.

Proof. 1. Construct the $\triangle A'B'C'$ congruent to $\triangle ABC$, by making $B'C' = BC$, $\angle B' = \angle B$, and $\angle C' = \angle C$. a. s. a.

2. Then $A'B'$ may be made to coincide with AB.
(Homologous sides of congruent ▵.)

3. But $\triangle A'B'C'$ may be made to coincide with $\triangle ABC$, even when turned over and placed so that B' falls on C and C' falls on B. a. s. a.

4. Then $A'B'$ will coincide with AC. Why?

5. $\therefore AB = AC$. Ax. 1
(Each being equal to $A'B'$.)

141. Corollary. *An equiangular triangle is also equilateral.*

Proposition IV is the **converse** of Proposition III.

The converse of a theorem is another theorem which results from interchanging the hypothesis and the conclusion of the given theorem.

Thus in Proposition III the

hypothesis is, *two sides of a triangle are equal;*
conclusion is, *the angles opposite those sides are equal.*

While in Proposition IV the

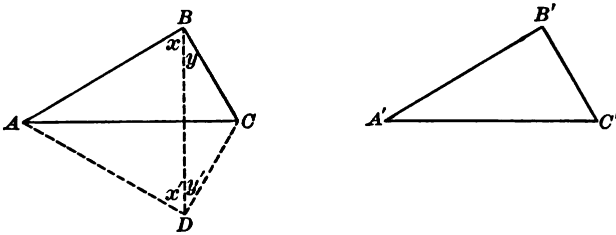
hypothesis is, *two angles of a triangle are equal;*
conclusion is, *the sides opposite those angles are equal.*

The converse of a proposition is not *necessarily* true, but must be *proved.*

Proposition V. Theorem

THIRD TRIANGLE CONGRUENCE

142. *If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are congruent.* (s. s. s.)



Given the triangles ABC and $A'B'C'$, in which AB equals $A'B'$, BC equals $B'C'$, and AC equals $A'C'$.

To prove that $\triangle ABC \equiv \triangle A'B'C'$.

Proof. 1. On that side of AC which is remote from B construct $\triangle ADC$ congruent to $\triangle A'B'C'$. a. s. a.

($AC = A'C'$, $\angle CAD = \angle A'$, and $\angle ACD = \angle C'$.)

Draw BD.

2. Then $\angle x = \angle x'$, and $\angle y = \angle y'$. § 138

($AB = AD$, and $CB = CD$, by hyp. and cons.)

3. $\therefore \angle ABC = \angle ADC$. Ax. 2

4. $\therefore \triangle ABC \equiv \triangle ADC$. s. a. s.

$\therefore \triangle ABC \equiv \triangle A'B'C'$. § 113

Discussion. Why cannot this proposition be proved directly by superposition? Note that in the figure it is tacitly assumed that $\angle BAC$ and $\angle BCA$ are both acute. What change would be necessary in the figure and proof if either of these angles were obtuse? Would a proof be necessary if either of these angles were a right angle?

143. Propositions III, IV, and V enable us to enlarge the scheme of § 136 as follows:

In order to prove two $\begin{pmatrix}\textbf{lines}\\ \textbf{angles}\end{pmatrix}$ **equal:**

I. Show that they are homologous $\begin{pmatrix}\textbf{sides}\\ \textbf{angles}\end{pmatrix}$ **of congruent** $\triangle$**s.**

In order to prove two $\triangle$*s congruent:*

1. Show that they have the relation a. s. a.
2. Show that they have the relation s. a. s.
3. Show that they have the relation s. s. s.

II. Show that in a $\triangle$ **they are** $\begin{pmatrix}\textbf{sides opposite equal angles}\\ \textbf{angles opposite equal sides}\end{pmatrix}$.

EXERCISES

1. Given, in the quadrilateral $ABCD$, $AD = BC$, $AB = CD$. Prove $\angle A = \angle C$. What construction line must first be drawn?

What other angles may also be proved equal? State as a general theorem.

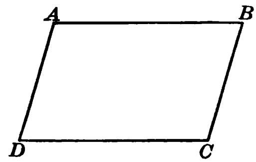


FIG. 1

2. Given, in the isosceles triangle ABC (Fig. 2 below), that $BD = CE$. Prove that $AD = AE$. State as a general theorem.

3. Given, in Fig. 3 below, that $AB = CD$, $AD = BC$, $BO = OD$. Prove that $EO = OF$. State as a general theorem.

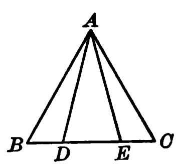


FIG. 2

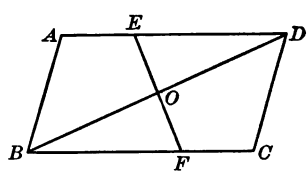


FIG. 3

4. Prove that if the diagonals of a quadrilateral bisect each other, the opposite sides of the quadrilateral are equal.

5. Prove that if the opposite sides of a quadrilateral are equal, the diagonals bisect each other.

6. Given, in Fig. 1, that $\triangle ABC$ is equilateral, and that $AD = BE = CF$. Prove that $\triangle DEF$ is also equilateral.

7. Given, in Fig. 2 below, that $\triangle ABC$ is equilateral, with its sides produced in succession so that $AD = BE = CF$. Prove that $\triangle DEF$ is also equilateral.

8. Given, in Fig. 3 below, that $\triangle ABC$ is equilateral, and that $\angle m = \angle n = \angle o$. Prove that $\triangle DEF$ is also equilateral.

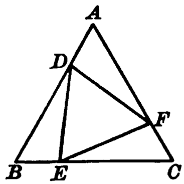


FIG. 1

9. Given, in Fig. 4 below, that $\triangle ABC$ is equilateral, and that $AF = BD = CE$. Prove that $\triangle HKL$ is also equilateral.

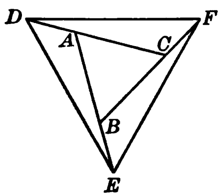


FIG. 2

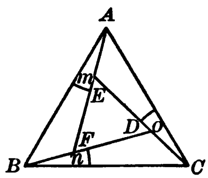


FIG. 3

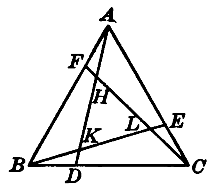


FIG. 4

144. Congruence of Polygons. The congruence of two polygons can be proved by methods similar to those of Propositions I and II. A polygon of four sides has eight parts, one of five sides ten parts, and, in general, one of n sides has $2n$ parts, n sides and n angles. It can be proved by superposition that if two polygons of n sides have $2n - 3$ *consecutive* parts of one respectively equal to $2n - 3$ consecutive parts of the other, the polygons are congruent. The abbreviations of Propositions I and II may be extended to indicate the congruence of two polygons, thus:

Two quadrilaterals are congruent if they have the relation a. s. a. s. a., *or the relation* s. a. s. a. s.

EXERCISES

1. In Ex. 3, p. 82, prove that the quadrilaterals $ABOE$ and $CDOF$ are congruent.

2. In Fig. 4, above, name all congruent quadrilaterals.

CONSTRUCTIONS

145. In order to show the equality of two lines or two angles in a figure, it is often necessary to introduce additional elements (points, lines, and circles) in such a way as to form two triangles which may be proved congruent.

The process by which these elements are introduced into a figure is called *construction.*

146. There are in plane geometry **three* fundamental constructions** upon which all other processes of construction are based. These have already been used repeatedly. They may be stated as follows:

1. *Joining any two points by a straight line.*
2. *Producing any straight line (segment) indefinitely.*
3. *Describing a circle about any point as a center, and with a radius equal to a given line (segment).*

147. These constructions often make possible the determination of points in a figure

1. As the intersection of two lines (one point).
2. As the intersection of a line and a circle (two points).
3. As the intersection of two circles (two points).

148. Several constructions have been shown to result directly:

1. Laying off on a straight line a length equal to a given line (segment).
2. Drawing a line so that it will form with a given line, at a given point, an angle equal to a given angle (§ 89).
3. The three fundamental triangle constructions (§§ 90, 91, 74).

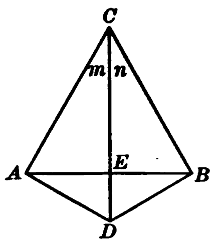
149. In establishing a construction it is necessary not only to describe the process in detail, but also to justify it; that is, to demonstrate that the steps in the process lead to the desired result.

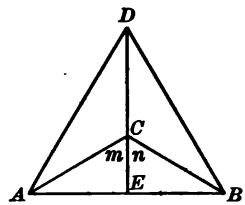
The next theorem is of value in proving some of the most important constructions.

* Since the time of Plato (about 390 B.C.) it has been customary in elementary geometry not to allow any other instruments than the straightedge and the compasses.

Proposition VI. Theorem

150. *If two isosceles triangles are constructed on the same base, the line that joins their vertices bisects the common base at right angles.*





Given two isosceles triangles *ABC* and *ABD*, on the common base *AB*, and the line *CD* cutting *AB* at *E*.

To prove that CE *bisects* AB *at right angles.*

Proof. 1.	$\triangle ACD \equiv \triangle BCD.$	s. s. s.
For	$AC = BC$, and $AD = BD$,	Hyp.
and	CD is common.	
2.	$\therefore \angle m = \angle n.$	Why?
3. Also	$\triangle ACE \equiv \triangle BCE.$	Why?
4.	$\therefore AE = BE,$	
and	$\angle AEC = \angle BEC.$	Why?
5.	$\therefore CE$ bisects AB at right angles.	

(Each angle at E being one half of a st. $\angle$.)

151. Corollary 1. *Two points each equidistant from the extremities of a line determine the perpendicular bisector of that line.*

A quadrilateral (such as $ACBD$ in the first of the above figures) which has two pairs of adjoining sides equal is called a **kite**.

152. Corollary 2. *The diagonals of a kite are perpendicular to each other.*

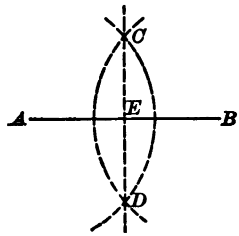
153. In order to prove that two lines are mutually $\perp$:

1. Show that they form equal supplementary-adjacent angles.

2. Show that they are diagonals of a kite.

Proposition VII. Problem

154. *To bisect a given straight line (line-segment).*



Given the line *AB*.

Required *to bisect* AB.

Construction. 1. With A and B as centers, and with equal radii sufficiently great, describe arcs intersecting at C and D.

2. Draw CD, cutting AB at E. Then E is the mid-point of AB.

Proof. 1. Draw AC, BC, AD, BD.

(To be completed, § 151.)

155. The line from any vertex of a triangle to the mid-point of the opposite side is called the **median** to that side.

EXERCISES

1. Explain how to divide a given line-segment into 4 (8, 16, 2^n) equal parts.

2. How many medians has a triangle? Draw a scalene triangle (§ 75) and construct its medians.

3. Draw a scalene triangle and construct the perpendicular bisectors of its sides.

4. Prove that the median to the base of an isosceles triangle is the perpendicular bisector of the base.

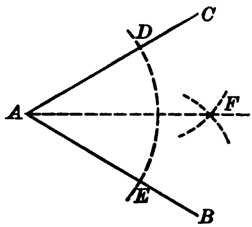
5. Prove that if the median of a triangle is perpendicular to the base, the triangle is isosceles.

6. Prove that the medians to the legs of an isosceles triangle are equal.

7. Prove that homologous medians of congruent triangles are equal.

Proposition VIII. Problem

156. *To bisect a given angle.*



Given the angle *CAB*.

***Required** to bisect $\angle CAB$.*

Construction. 1. With A as a center, and with any convenient radius, as AD, describe an arc cutting AC in D, and AB in E.

2. With D and E as centers, and with equal radii sufficiently great, describe arcs intersecting in F. Draw AF. Then AF bisects $\angle A$.

Proof. 1. Draw DF and EF.

(To be completed, § 142.)

157. Corollary. *To bisect a given arc of a given circle,* bisect the central angle intercepted by the arc. The line which bisects this angle will, if produced, bisect the given arc also. Why?

EXERCISES

1. Construct a scalene triangle and draw the bisectors of the interior angles, producing each to meet the opposite side.

2. Prove that the bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.

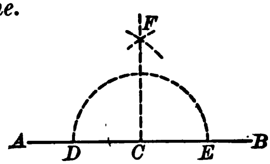
3. Prove that the bisectors of the base angles of an isosceles triangle, terminating in the opposite sides, are equal.

4. How can an angle be bisected by the use of a carpenter's square?

5. Prove that two triangles are congruent if they have a side, an adjoining angle, and the bisector of that angle in one equal respectively to the corresponding parts in the other.

PROPOSITION IX. PROBLEM

158. *At a given point in a given line, to erect a perpendicular to that line.*



Given the point *C* in the line *AB*.

Required *to erect a perpendicular to AB at C.*

Construction. 1. Lay off $CD = CE$.

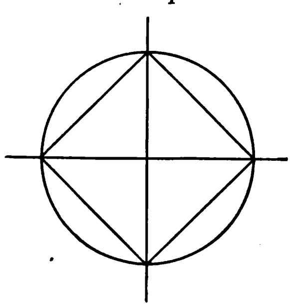
2. With D and E as centers, and with equal radii sufficiently great, describe arcs intersecting at F. Draw FC. Then FC is the perpendicular required.

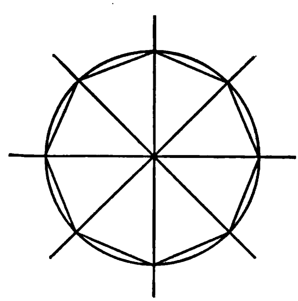
Proof. Draw FD and FE.

(To be completed, § 142.)

EXERCISES

1. Draw two mutually perpendicular lines. On the four rays lay off equal segments from the point of intersection, and connect their extremities. Prove that the sides and the angles of the resulting quadrilateral are equal.

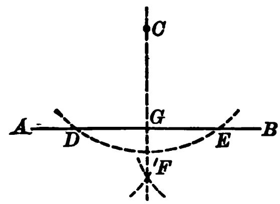




2. Bisect a straight angle and bisect both halves, producing each bisector in turn through the vertex. Then proceed as in Ex. 1. What kind of polygon is formed? (See § 57.)

PROPOSITION X. PROBLEM

159. *From a given point without a given line, to let fall a perpendicular upon that line.*



Given the line *AB* and the point *C* without it.

Required *to let fall a perpendicular from C to AB.*

Construction. 1. With C as a center, and with a radius sufficiently great, describe an arc cutting AB in D and E.

2. With D and E as centers, and with equal radii sufficiently great, describe two arcs intersecting at F. Draw CF (producing it if necessary) cutting AB in G. Then CG is the perpendicular required.

Proof. Draw CD and CE, also FD and FE.

(To be completed, § 150.)

160. An **altitude** of a triangle is a perpendicular let fall from any vertex to the side opposite, produced if necessary. The opposite side is called the **corresponding base.**

EXERCISES

1. How many altitudes has a triangle?

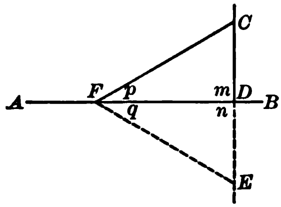
2. Construct the altitudes of a scalene acute triangle; of an equilateral triangle; of an isosceles acute triangle.

3. Construct the altitudes of an obtuse triangle. What additional construction lines are necessary?

NOTE. The construction given above is strictly geometric (see footnote, p. 84). In practice a perpendicular is drawn by the use of a model of a right angle, such as a carpenter's square or a draftsman's triangle (see p. 103).

PROPOSITION XI. THEOREM

161. *Of all lines that can be drawn from a given point to a given line, only one is perpendicular to the given line.*



Given the line *AB*, the point *C* without it, and *CD* perpendicular to *AB*.

To prove *that any other line, as CF, drawn from C to AB is not perpendicular to AB.*

Proof. 1. Produce CD to E, making $DE = CD$, and draw EF.

2. Then $\triangle CDF \equiv \triangle EDF$. s. a. s.

For DF is common,

$CD = DE$, Cons.

and $\angle m = \angle n$. Why?

3. $\therefore \angle p = \angle q$.

(Homologous $\angle$s of congruent $\triangle$s.)

4. Now CDE is a straight line. Cons.

$\therefore$ CFE is not a straight line. §11

$\therefore \angle CFE$ is not a straight angle.

5. $\therefore \angle p$ (one half of $\angle CFE$) is not a rt. $\angle$.

6. $\therefore$ CF is not perpendicular to AB, nor is any other line from C perpendicular to AB except CD.

162. COROLLARY. *A triangle can have but one right angle.*

RIGHT TRIANGLES

Construction of Right Triangles

163. By applying the methods of §§ 89, 158, 159 it is possible to construct a right triangle, having given

1. A leg and the adjoining oblique angle.
2. The two legs.
3. The hypotenuse and a leg.
4. The hypotenuse and an adjoining angle.

These constructions correspond to as many special laws of

Congruence of Right Triangles

164. From the First and Second Triangle Congruences it follows that

1. *If two right triangles have a leg and the adjoining oblique angle of one equal respectively to a leg and the adjoining oblique angle of the other, the triangles are congruent.* (a. s. a.)

2. *If two right triangles have two legs of one equal respectively to two legs of the other, they are congruent.* (s. a. s.)

EXERCISES

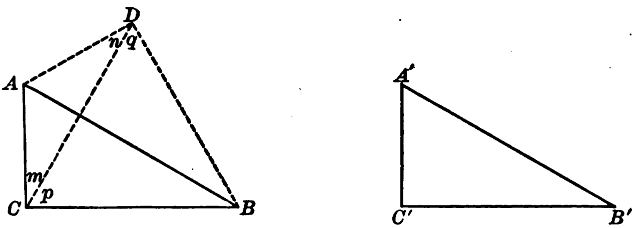
1. Prove that if the altitude of a triangle bisects the base, the triangle is isosceles.

2. Prove that if the altitude of a triangle bisects the vertex angle, the triangle is isosceles.

3. Roman surveyors, called *agrimensores*, are said to have used the following method of measuring the width of a stream: A and B were points on opposite sides of the stream, in plain view from each other. The distance AD was then taken at right angles to AB, and bisected at E. Then the distance DF was taken at right angles to AD, such that the points B, E, and F were in a straight line. Make a drawing illustrating the above method, show what measurement affords a solution, and prove that this is so.

Proposition XII. Theorem

165. *If two right triangles have the hypotenuse and a leg of one equal respectively to the hypotenuse and a leg of the other, the triangles are congruent.* (rt. △ h. l.)



Given the right triangles *ABC* and *A'B'C'*, with the hypotenuse *AB* equal to the hypotenuse *A'B'*, and with *AC* equal to *A'C'*.

***To prove* that** $\quad rt. \triangle ABC \equiv rt. \triangle A'B'C'$.

Proof. 1. On AB ($= A'B'$) construct rt. $\triangle ADB$ congruent to rt. $\triangle A'B'C'$, so that $AD = A'C'$, and $BD = B'C'$. Draw CD.

2. Then $\quad \angle m = \angle n. \quad$ § 138

(Since $AC = AD$, by hyp. and cons.)

3. Now $\quad \angle ACB = \angle ADB. \quad$ Why?

$\therefore \angle p = \angle q, \quad$ Ax. 3

and hence $\quad BC = BD. \quad$ Why?

4. $\quad \therefore \triangle ACB \equiv \triangle ADB, \quad$ s. s. s.

that is, $\quad \triangle ABC \equiv \triangle A'B'C'. \quad$ Why?

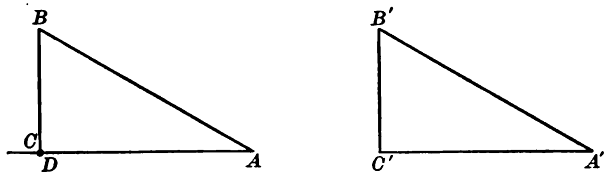
EXERCISES

1. Prove that a ladder of known length, the foot of which is at a known horizontal distance from the wall of a house, always reaches the same distance up the side of the house.

2. Prove that the altitude to the base of an isosceles triangle bisects the base and the vertex angle.

PROPOSITION XIII. THEOREM

166. *If two right triangles have the hypotenuse and an adjoining angle of one equal respectively to the hypotenuse and an adjoining angle of the other, the triangles are congruent.* (rt. △ h. a.)



Given the right triangles *ABC* and *A'B'C'*, with the hypotenuse *AB* equal to the hypotenuse *A'B'*, and with the angle *A* equal to the angle *A'*.

***To prove* that** $\quad rt. \triangle ABC \equiv rt. \triangle A'B'C'$.

Proof. 1. On AC take $AD = A'C'$, and draw BD.

2. Then	$\triangle ABD \equiv \triangle A'B'C'$.	s. a. s.
3. Hence	$\angle ADB$ ($= \angle A'C'B'$) is a rt. $\angle$.	Why?
4.	$\therefore BD$ coincides with BC.	§ 161
5.	$\therefore \triangle ABD \equiv \triangle ABC$;	Def.
that is,	$\triangle ABC \equiv \triangle A'B'C'$.	Why?

167. A further enlargement of the scheme of § 143 is now possible.

In order to prove two (lines / angles) equal:

I. Show that they are homologous (sides / angles) of congruent △.

In order to prove two triangles congruent:

1. Show that they have the relation a. s. a.
2. Show that they have the relation s. a. s.
3. Show that they have the relation s. s. s.
4. Show that they have the relation rt. △ h. l.
5. Show that they have the relation rt. △ h. a.

II. Show that in a △ they are (sides opposite equal angles / angles opposite equal sides).

EXERCISES

1. Prove that the perpendiculars let fall from the mid-points of the legs of an isosceles triangle upon the base are equal.

2. Prove that the altitudes on the legs of an isosceles triangle are equal.

3. Prove that homologous altitudes of congruent triangles are equal.

4. Prove that the bisectors of homologous angles of congruent triangles are equal.

5. In transferring a line-segment by means of the dividers, which law of congruence is applied? Is it necessary that the legs of the dividers should be of equal length?

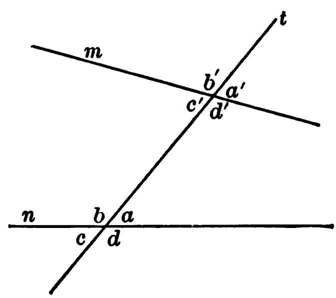
6. By what measurements can a carpenter ascertain that the two gable triangles of a roof are exactly alike?

7. Show how, by measuring sides and diagonals, a surveyor may determine completely the size and shape of an irregular field, without measuring any angles. (Assume that the boundaries of the field are straight lines.)

8. State the above problem in the form of a geometric theorem in the comparison of polygons.

PARALLEL LINES

168. A line cutting two or more lines is a **transversal** of these lines. The transversal t in the figure forms with the lines m and n angles that are classified as follows:



a, b, c', d' are **interior angles**;

a', b', c, d are **exterior angles**;

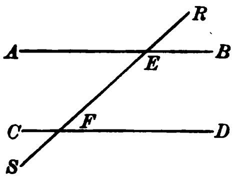
a and c' are **alternate-interior angles**, also b and d';

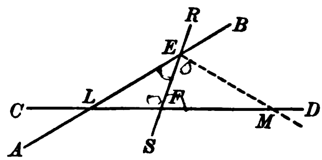
a' and c are **alternate-exterior angles**, also b' and d;

a' and a are **exterior-interior** or **corresponding angles**, also b' and b, c' and c, d' and d.

Proposition XIV. Theorem

169. *When two straight lines are cut by a transversal, if the alternate-interior angles are equal, the two straight lines do not intersect.*





Given two straight lines *AB* and *CD* cut by the transversal *RS*, with the angle *AEF* equal to the angle *EFD*.

To prove that *AB cannot intersect CD.*

Proof. 1. If possible, let AB intersect CD at L. Take $FM = EL$, and draw EM.

2. Then	$\triangle EFL \equiv \triangle EFM.$	s. a. s.
For	$EL = FM,$	Cons.
	EF is common,	
and	$\angle EFM = \angle FEL.$	Hyp.
3.	$\therefore \angle EFL = \angle FEM.$	Why?
4. But	$\angle EFL + \angle EFM =$ a st. $\angle$.	Why?
5. Hence	$\angle FEM + \angle FEL =$ a st. $\angle$.	Ax. 1

6. Hence LEM would be a straight line cutting CD in two points L and M.

7. Since that is impossible (why?),

AB cannot intersect CD.

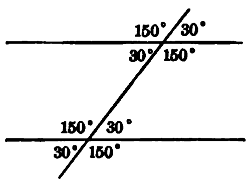
(The method of proof used here is called the **Indirect Method.**)

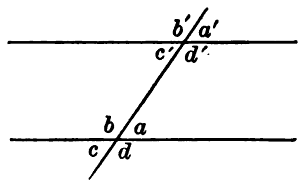
170. Parallel lines are lines which lie in the same plane and do not meet, however far they are produced.

Thus in the above theorem AB is parallel to CD. This is often written $AB \parallel CD$.

Proposition XIV may now be stated: *When two straight lines are cut by a transversal, if the alternate-interior angles are equal, the straight lines are parallel.*

The figures below show that if the alternate-interior angles are equal, then the alternate-exterior angles are also equal; and if the corresponding angles are equal, the two sets of alternate angles are equal, etc. In fact, if any one of the following twelve equations is true, they are all true.





$\angle a' = \angle c$, $\angle b' = \angle d$, $\angle a = \angle c'$, $\angle b = \angle d'$ — alternate angles.

$\angle a = \angle a'$, $\angle b = \angle b'$, $\angle c = \angle c'$, $\angle d = \angle d'$ — corresponding angles.

$\angle a' + \angle d =$ a st. $\angle$.

$\angle b' + \angle c =$ a st. $\angle$.

$\angle a + \angle d' =$ a st. $\angle$.

$\angle b + \angle c' =$ a st. $\angle$.

Proposition XV. Theorem

171. *When two straight lines are cut by a transversal, if*

(*a*) *the alternate-exterior angles are equal; or if*

(*b*) *the exterior-interior angles are equal; or if*

(*c*) *any two interior or any two exterior angles on the same side of the transversal are supplementary,*

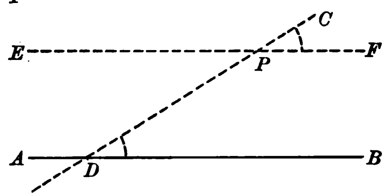
the two straight lines are parallel.

For any one of these relations may be reduced to the equality of alternate-interior angles.

172. Corollary. *Two straight lines in the same plane perpendicular to the same straight line are parallel.*

Proposition XVI. Problem

173. *To construct a line parallel to a given line, through a given external point.*



Given a straight line *AB* and the external point *P*.

Required *to construct through P a line parallel to AB.*

Construction. 1. Through P draw a transversal cutting AB at D.

2. Construct $\angle CPF = \angle CDB$. Then the line EPF is $\parallel$ to AB. § 171

(What other angle-pairs might have been used for this construction ?)

174. We now assume the following

Parallel Axiom. *Through a given point but one straight line can be drawn parallel to a given straight line.*

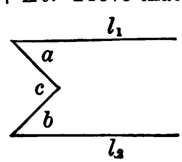
This assumption may also be stated as follows:

Two intersecting straight lines cannot both be parallel to the same straight line.

175. Corollary. *Two straight lines in the same plane parallel to the same straight line are parallel to each other.*

EXERCISE

1. Given in the annexed figure that $\angle c = \angle a + \angle b$. Prove that $l_1 \parallel l_2$.



Suggestion. Through the vertex of $\angle c$ draw a line l_3 dividing $\angle c$ into $\angle x$ and $\angle y$ and making $\angle x = \angle a$. Then $l_1 \parallel l_3$. (Why?) Also $l_2 \parallel l_3$. (Why?) $\therefore l_1 \parallel l_2$. (Why?)

Proposition XVII. Theorem

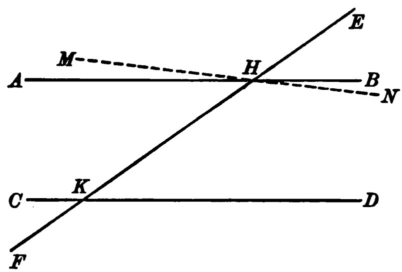
176. *If two parallel lines are cut by a transversal:*

(a) *the alternate-interior angles are equal;*

(b) *the exterior-interior angles are equal;*

(c) *the alternate-exterior angles are equal;*

(d) *any two interior or any two exterior angles on the same side of the transversal are supplementary.*



Given the parallel lines *AB* and *CD* cut by the transversal *EF* at *H* and *K*.

To prove (a) *that* $\angle AHK = \angle HKD$.

Proof. 1. If $\angle AHK$ is not equal to $\angle HKD$, draw MN through H, making $\angle MHK = \angle HKD$.

2. Then $MN \parallel CD$. § 170

(Since the alt.-int. ∠s are equal.)

3. But $AB \parallel CD$. Hyp.

Then MN and AB are *two* lines drawn through the same point H, and parallel to CD.

4. But this is impossible. § 174

5. Hence MN must coincide with AB.

$\therefore \angle MHK = \angle AHK = \angle HKD$. Why?

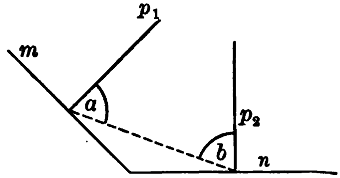
Conclusions (b), (c), (d) follow directly from this.

(To be completed.)

(This theorem is the converse of Proposition XV.)

177. Corollary 1. *A straight line perpendicular to one of two parallel lines is perpendicular to the other also.*

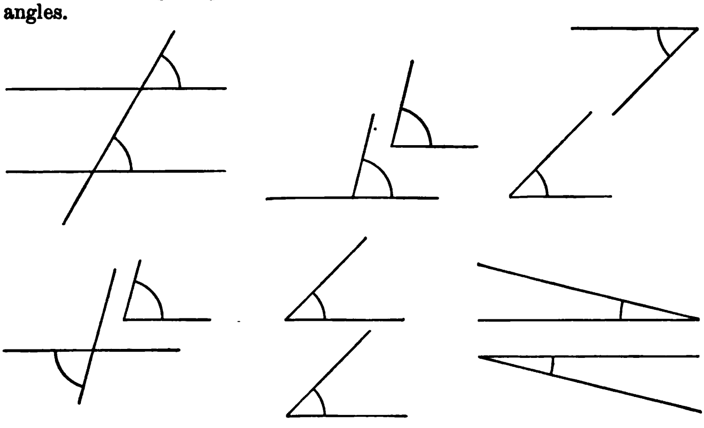
178. Corollary 2. *If two lines are perpendicular respectively to two intersecting lines, they cannot be parallel.*



If $p_1 \perp m$, and $p_2 \perp n$, then $\angle a <$ rt. $\angle$, and $\angle b <$ rt. $\angle$. $\therefore p_1$ is not parallel to p_2, since $\angle a + \angle b <$ a st. $\angle$.

179. Corollary 3. *If two angles have their sides respectively parallel, they are equal or supplementary.*

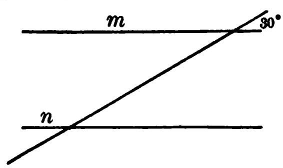
The following diagrams illustrate various relative positions of such angles.



EXERCISES

1. Give ten concrete illustrations of parallels.

2. In the figure, $m \parallel n$, and one of the exterior angles is 30°. Find the values of the other angles.

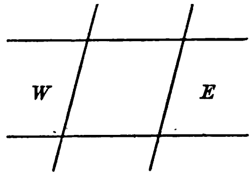


3. How many sets of numerically different angles does the figure (Ex. 2) contain?

4. If the exterior angles (Ex. 2) are in the ratio 1 : 2 (2 : 3 ; 4 : 5), determine their values.

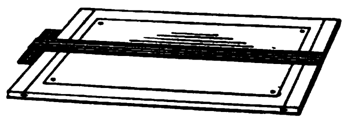
5. If the ratio of the angles in Ex. 4 is 1 : 1, what is the position of the transversal with reference to the parallels?

6. Let the figure represent two intersecting streets. How many numerically different angles are there?

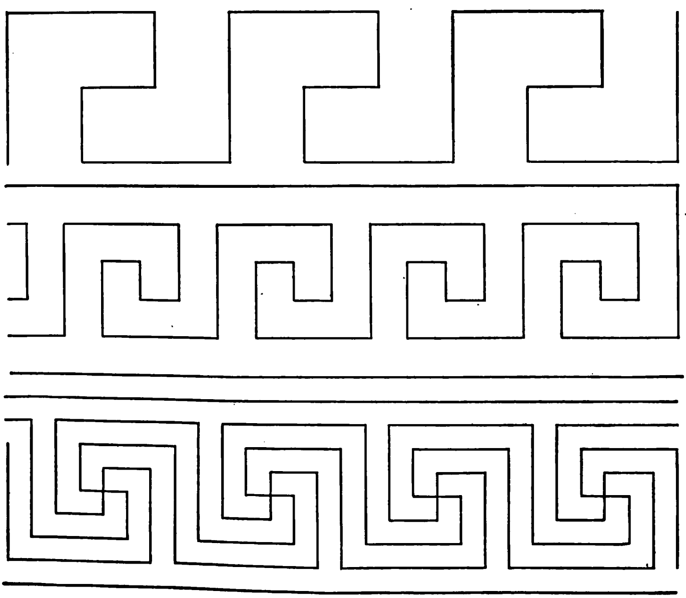


7. If one of the streets extends in an east-and-west direction, and the ratio of the angles is 1 : 3, what is the direction of the other street?

8. The T-square is an instrument constantly used in mechanical drawing (see illustration). By its means a series of parallel lines may be drawn at any desired intervals. Upon what theorem of parallels does its use depend?



9. Copy the following Greek designs, called meanders. What method of constructing parallels is used?



10. Which laws of parallels are illustrated by the following capital letters: Z, N, H, E, F, W?

11. In Fig. 1, $l_1 \parallel l_2$; prove that $\angle c = \angle a + \angle b$.

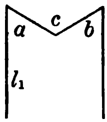


FIG. 1

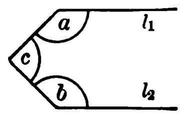


FIG. 2

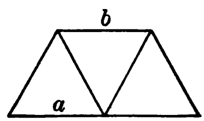
12. In Fig. 2, $l_1 \parallel l_2$; prove that $\angle a + \angle b + \angle c = 2$ st. $\angle$s.

13. If l_1, l_2, etc. are different lines, construct the following figures:

(1) $l_1 \parallel l_2$, $l_1 \parallel l_3$;
(2) $l_1 \parallel l_2$, $l_2 \parallel l_3$;
(3) $l_1 \perp l_2$, $l_2 \perp l_3$;
(4) $l_1 \perp l_2$, $l_2 \parallel l_3$;
(5) $l_1 \perp l_2$, $l_2 \parallel l_3$, $l_3 \perp l_4$;
(6) $l_1 \parallel l_2$, $l_2 \perp l_3$, $l_3 \parallel l_4$.

In (1) what is the position of l_2 with reference to l_3? in (2)–(6) what are the relative positions of the first line and the last line?

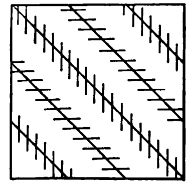
14. Construct three equilateral triangles in the positions shown in the figure. Prove that $b \parallel a$. What other lines are parallel?



15. The following is a practical method of drawing through a given point a line parallel to a given line. Perform the construction as indicated, and prove that it leads to the desired result.

Given the line BC and the point A. Required to draw through A a line parallel to BC.

With A as a center and a radius sufficiently great describe a circle cutting the line BC in the point D. Draw AD. On BC lay off DE equal to AD, and in the circle draw a chord DF equal to the distance AE, so that DF and AE lie on the same side of BC, but on opposite sides of AD. Draw AF and produce it if necessary. AF is parallel to BC.

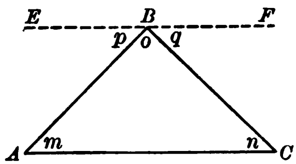


16. In the accompanying figure there are four lines which are drawn parallel to the diagonal of the square. Do they *appear* to be parallel to the diagonal and to one another?

ANGLE-SUM

Proposition XVIII. Theorem

180. *The sum of the interior angles of a triangle equals a straight angle.*



Given the triangle *ABC*, whose angles are *m*, *n*, and *o*.

To prove *that* $\angle m + \angle n + \angle o = a \text{ st. } \angle.$

Proof. 1. Through B draw $EF \parallel AC$.

2. Then $\angle m = \angle p,$ § 176

and $\angle n = \angle q.$ Why?

3. $\therefore \angle m + \angle o + \angle n = \angle p + \angle o + \angle q.$ Ax. 2

4. But $\angle p + \angle o + \angle q = \text{a st. } \angle.$ Why?

$\therefore \angle m + \angle o + \angle n = \text{a st. } \angle.$ Ax. 1

181. Corollary 1. *An exterior angle of a triangle is equal to the sum of the two remote interior angles, and is therefore greater than either of them.*

182. Corollary 2. *If two angles of one triangle are equal respectively to two angles of another triangle, the third angles are equal.*

183. Corollary 3. *Two triangles are congruent if a side and any two angles of one are equal respectively to a side and two angles, similarly situated, of the other.*

184. Corollary 4. *The sum of two angles of a triangle is less than a straight angle.*

185. Corollary 5. *If a triangle has one right angle or one obtuse angle, the other angles are acute.*

186. Corollary 6. *Every triangle has at least two acute angles.*

187. Corollary 7. *The sum of the two acute angles of a right triangle equals a right angle.*

188. Corollary 8. *Each angle of an equilateral triangle is an angle of 60°.*

189. Corollary 9. *If one leg of a right triangle is half the hypotenuse, then the angle opposite that leg is an angle of 30°, and the other acute angle is an angle of 60°.*

190. Corollary 10. *If a right triangle contains acute angles of 30° and 60° respectively, then the leg opposite the angle of 30° is one half the hypotenuse.*

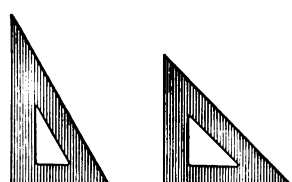
191. Corollary 11. *If the legs of a right triangle are equal, each acute angle is an angle of 45°.*

192. Corollary 12. *If one acute angle of a right triangle is an angle of 45°, then the other acute angle is also an angle of 45°, and the legs of the right triangle are equal.*

193. Corollary 13. *Each base angle of an isosceles triangle is half the supplement of the vertex angle.*

EXERCISES

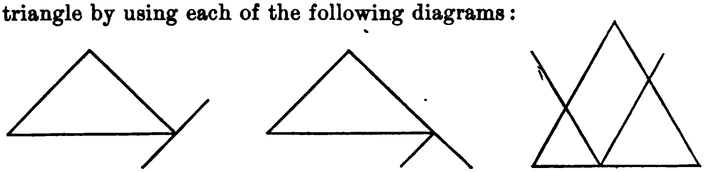
1. In mechanical drawing constant use is made of certain fixed right triangles which are made of wood or celluloid. One of these has acute angles of 30° and 60°, while the other has acute angles of 45° each. What angles can a draftsman draw with the aid of these two triangles, but without bisecting any angle?



2. Show how one of the above triangles, combined either with the other triangle or with the T-square (see Ex. 8, p. 100), may be used to draw a series of parallels.

EXERCISES

1. Prove the proposition concerning the sum of the angles of a triangle by using each of the following diagrams:



2. Can a triangle be formed with the following angles: (1) 40°, 50°, 90°? (2) 30°, 20°, 50°? (3) 70°, 80°, 100°? (4) 100°, 200°, 300°? (5) 10°, 20°, 30°? (6) $75\frac{1}{2}°$, $4\frac{1}{3}°$, $100\frac{1}{6}°$? (7) $(90-a)°$, $(90+a)°$, $b°$? (8) $(90-x)°$, $(90+y)°$, $(x-y)°$? (9) 76° 31′ 10″, 94° 18′ 6″, 9° 10′ 45″?

3. Find the third angle c, if two angles of a triangle, a and b, have the following values:

	(1)	(2)	(3)	(4)	(5)	(6)
	$a = 20°$	60°	100°	45°	$72\frac{1}{2}°$	72°
	$b = 30°$	70°	79°	45°	45°	36°

4. Determine the values of the angles a, b, c of a triangle, if (1) their ratio is 1 : 2 : 3; (2) $a = b$, $b = \frac{1}{2}c$; (3) $a = b + 20°$, $b = c + 20°$; (4) $a + b = 70°$, $b + c = 150°$; (5) their ratio is 1 : 1 : 2.

5. Find the value of each angle of an isosceles triangle, if the vertex angle contains 30°; 40°; 50°; $\frac{3}{4}$ rt. ∠; 60°; 70°; $\frac{4}{5}$ rt. ∠; 80°; 100°; $\frac{6}{5}$ rt. ∠; 120°; 130°; 150°; 42° 16′; 16° 28′ 52.″

6. Find the value of each angle of an isosceles triangle, if one of the base angles contains 30°; 35°; 40°; 45°; 50°; $\frac{2}{9}$ rt. ∠; 55°; $\frac{3}{8}$ st. ∠; 80°; 89°; 57° 43′; 68° 51′ 7″.

7. One acute angle of a right triangle is 30°; 40°; 45°; $48\frac{1}{2}°$; 60°; 62°; $42\frac{1}{2}°$; $22\frac{1}{2}°$. Find the value of the other acute angle and of each of the exterior angles.

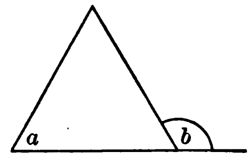
8. How many degrees in each exterior angle of an equilateral triangle?

9. In what kind of triangle is one angle sufficient for the determination of the other two?

10. If the vertex angle of an isosceles triangle contains 120°, show that the altitude divides the triangle into two triangles that can be placed so as to form an equilateral triangle.

11. What is the resulting figure in Ex. 10 if the vertex angle is a right angle?

12. The base angles of an isosceles triangle are each 40°. Find the angle formed by their bisectors; also the angle formed by the bisectors of their exterior angles.



13. In the figure $\angle a + \angle b =$ a st. $\angle$. What kind of triangle is it?

14. Each angle of a triangle is the supplement of the sum of the other two.

15. Two angles of a triangle cannot be supplementary.

16. The base angles of an isosceles triangle are acute.

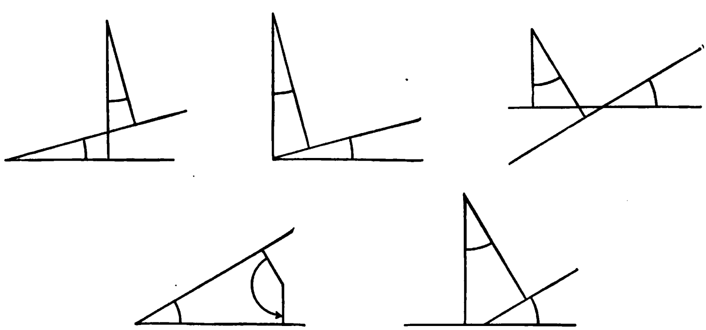
17. An isosceles triangle is obtuse, right, or acute, according as a base angle is less than, equal to, or greater than 45°

18. If an angle of an isosceles triangle is obtuse, it is the vertex angle.

19. The sum of the acute angles of an obtuse triangle is an acute angle.

20. The bisectors of the base angles of an isosceles triangle form with the base an isosceles triangle whose vertex angle is the supplement of each base angle of the first triangle.

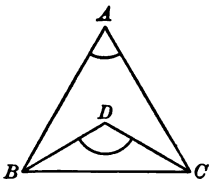
21. If the sides of one angle are perpendicular to the sides of another angle, the angles are equal or supplementary.

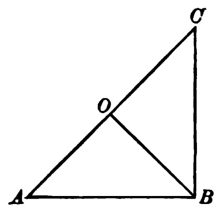


22. If a perpendicular is dropped from the vertex of the right angle of a right triangle to the hypotenuse, prove that the two triangles thus formed and the given triangle are mutually equiangular.

23. The perpendiculars dropped from any point in the bisector of an angle to its sides make equal angles with the bisector.

24. In the first figure below prove that $\angle D$ is greater than $\angle A$.

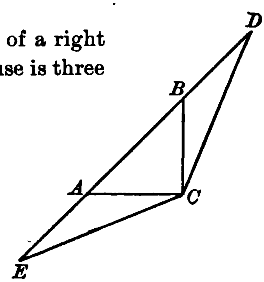




25. If one of the equal sides OA of the isosceles $\triangle OAB$ (figure above) is produced through the vertex O to C, so that $OC = OA$, prove that $\triangle ABC$ is a right triangle. What construction does this suggest?

26. The bisectors of the acute angles of a right triangle form an angle of 135°.

27. The sum of the two exterior angles of a right triangle formed by extending the hypotenuse is three right angles.

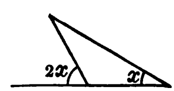


28. The hypotenuse AB of a right triangle ABC is extended in both directions so that $BD = BC$, and $AE = AC$. Prove that $\angle DCE$ contains 135°.

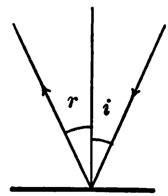
29. Given two angles of a triangle, to construct the third angle.

30. Given a line l and two exterior points A and B. From A and B drop perpendiculars AC and BD to l. Connect A and B. Prove that $\angle A = \angle B$ if A and B are on opposite sides of l; and that $\angle A + \angle B =$ a st. $\angle$ if they are on the same side of l.

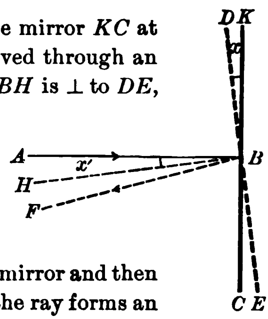
31. A sailor often measures his distance from a lighthouse by "doubling the angle on the bow." He notes the angle which the line of the lighthouse makes with the course of the vessel at a certain time, and again he notes the time at which the angle is exactly doubled. The distance the vessel has traveled in the interval can be determined from its rate. Show that this distance is equal to the distance of the vessel from the lighthouse at the time last taken. (From the National Syllabus of Geometry.)



32. When a ray of light strikes a plane mirror it is reflected in accordance with the following law: The angle between the incident ray and the perpendicular to the mirror is equal to the angle between the reflected ray and the same perpendicular, and both angles lie in the same plane. The first is called the angle of incidence, the second the angle of reflection. When a ray strikes the mirror perpendicularly, it is reflected back over the same path.



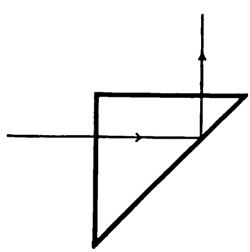
A ray of light AB is incident on a plane mirror KC at right angles to it. Let the mirror be revolved through an angle x, taking the new position DE. If BH is $\perp$ to DE, prove that $\angle x' = \angle x$, and that the angle between the incident ray AB and the reflected ray BF is equal to $2\angle x$. (This principle is of importance in measuring small movements of rotation.)



33. A ray of light is reflected first by one mirror and then by another in such a way that the path of the ray forms an equilateral triangle. What is the angle between the mirrors? (Assume that the mirrors are perpendicular to the plane of the triangle.)

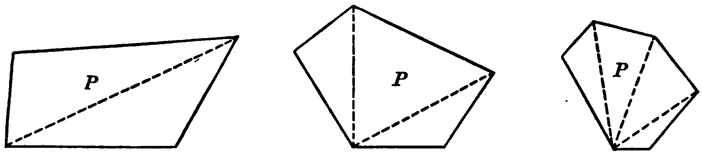
34. A ray of light is reflected by two mirrors successively in such a way that it returns along a line parallel to its original direction. What is the angle between the mirrors? Does this angle depend upon the direction of the transverse ray from one mirror to the other?

35. It is found that within certain limits a ray of light is totally reflected from the inner surface of glass. This property of glass is of advantage in the construction of optical instruments. The adjoining figure represents a right section of a prism, in the form of an isosceles right triangle. A ray of light enters the prism perpendicular to the side represented by one leg of the right triangle. In what direction does it leave the prism? (Assume, as is the case, that the ray is within the limits of total reflection.)



PROPOSITION XIX. THEOREM

194. *The sum of the interior angles of a polygon is equal to as many straight angles as the figure has sides less two.*



Given the polygon P of n sides.

To prove *that the sum of the interior angles of* $P = (n-2)$ *st.* ∠s.

Proof. 1. Draw all the diagonals from some one vertex.

There will be formed $n-2$ triangles. Why?

2. Now the sum of the angles of each triangle equals a st. ∠.

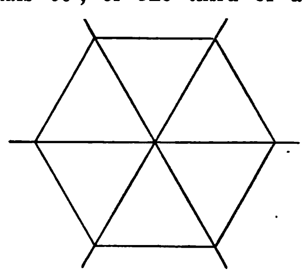
∴ the sum of the angles of all the triangles of the polygon equals $(n-2)$ st. ∠s. Why?

3. But the sum of the angles of all the triangles is the sum of the angles of the polygon.

∴ the sum of the angles of the polygon equals $(n-2)$ st. ∠s.

EXERCISES

1. It is now possible to construct a regular hexagon. Since each angle of an equilateral triangle equals 60°, or one third of a straight angle, such an angle applied consecutively to a straight angle divides it into three equal parts. If the lines of division are produced through the vertex, and equal distances are laid off on the six rays from the vertex, the lines joining in succession the points thus determined form a regular hexagon. Prove this.

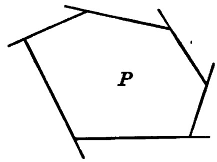


2. Show how to construct a regular dodecagon; a regular 24-gon.

3. What regular polygons can a draftsman draw with the aid of the fixed triangles shown on page 103, without bisecting angles.

Proposition XX. Theorem

195. *If the sides of a polygon are produced in succession, the sum of the exterior angles thus formed equals two straight angles.*



Given the polygon P of n sides, with its sides produced in succession.

To prove *that the sum of the exterior angles of P thus formed equals two straight angles.*

Proof. 1. P has n vertices. At each vertex the sum of the interior angle and its adjacent exterior angle equals a straight angle.

2. $\therefore$ the sum of the interior and exterior $\angle$s of $P = n$ st. $\angle$s. Why?

3. But the sum of the interior $\angle$s of $P = (n - 2)$ st. $\angle$s. Why?

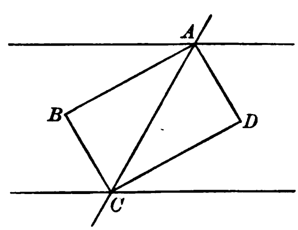
4. $\therefore$ the sum of the exterior $\angle$s of $P = 2$ st. $\angle$s. Ax. 3

EXERCISES

1. If the values of three angles of a quadrilateral are 70°, 80°, 100°, what is the value of the fourth angle?

2. Three angles of a quadrilateral are right angles. Find the fourth angle.

3. Prove that if two opposite angles of a quadrilateral are supplementary, the other two angles are also supplementary.



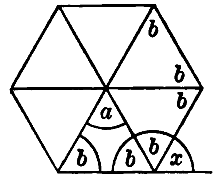
4. If two parallel lines are cut by a transversal, the bisectors of the interior angles are perpendicular to each other.

5. The bisectors of the base angles of an isosceles triangle and of the adjacent exterior angles form a quadrilateral in which the opposite angles are supplementary.

6. How many degrees in the sum of the interior angles of a polygon of 4 (5, 6, 7, 8, 9, 10) sides?

7. If these polygons are regular, how many degrees in each interior angle? in each exterior angle?

8. In a regular polygon a central angle is equal to an exterior angle.



Proof. $a + 2b = 180° = x + 2b$.

$\therefore x = a$.

9. Make a table giving the number of degrees in the central angles, the interior angles, and the exterior angles of the regular polygons known to you.

10. How may the number of sides of a regular polygon be determined, if we know the value of an exterior angle?

11. How many sides has a polygon, if

(a) The sum of the interior angles equals 4 rt. ∡? 3 st. ∡? 6 rt. ∡? 8 st. ∡? 20 rt. ∡?

(b) The sum of the interior angles is 2 (3, 4, 5, 6) times as large as the sum of the exterior angles?

(c) The sum of the interior angles exceeds the sum of the exterior angles by 4 rt. ∡? 3 st. ∡? 9 st. ∡?

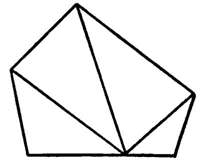
(d) The ratio of each interior angle to its adjacent exterior angle is 2 : 1? 3 : 2? 5 : 1? $a : b$?

(e) Each exterior angle contains 40°? 30°? 20°? 120°?

(f) Each interior angle is $\frac{1}{3}$ st. ∠? $\frac{3}{5}$ st. ∠? a rt. ∠? $\frac{3}{2}$ rt. ∠? $\frac{8}{5}$ rt. ∠? $\frac{7}{8}$ st. ∠?

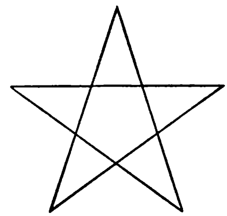
12. How does an increase in the number of sides of a regular polygon affect each interior angle? each exterior angle?

13. Prove the proposition concerning the sum of the angles of a polygon by drawing lines from any point in the perimeter, as in the figure.



14. The known relation between a central angle of a regular polygon and the base angles of one of the component isosceles triangles makes it possible to construct a regular polygon on a given side. Explain.

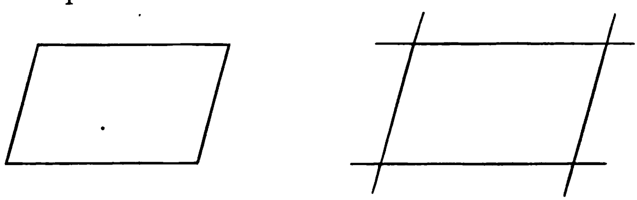
15. If the sides of a polygon are produced in both directions, sets of exterior angles are formed whose sum is 8 rt. $\angle$s.



16. If the sides of a polygon are extended until they intersect, a *star polygon* results. A five-pointed star is called a *pentagram*. It is of historic interest, having been chosen by the followers of Pythagoras as their badge. Star polygons may also be formed by chords of circles or by certain combinations of polygons. What is the sum of the vertex angles of a five-pointed star? a six-pointed star? an n-pointed star?

PARALLELOGRAMS

196. A **parallelogram** is the quadrilateral inclosed when two pairs of parallel lines intersect each other.



PRELIMINARY PROPOSITIONS

197. *Any two consecutive angles of a parallelogram are supplementary.*

198. *The opposite angles of a parallelogram are equal.*

199. *If one angle of a parallelogram is a right angle, the other angles are also right angles.*

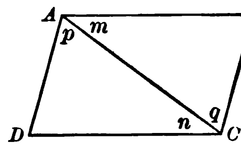
200. *If two parallelograms have two adjacent sides and the included angle of one equal respectively to the corresponding parts of the other, they are congruent.*

They are quadrilaterals, and it can be readily shown that they have the relation a. s. a. s. a. (§ 144).

Hence a parallelogram may be constructed when two sides and the included angle are given.

PROPOSITION XXI. THEOREM

201. *A diagonal of a parallelogram divides it into two congruent triangles, and the opposite sides of a parallelogram are equal.*



Given the parallelogram *ABCD*, with the diagonal *AC*.

To prove that $\triangle ABC \equiv \triangle ADC$, *and that* $AB = CD$, *and* $AD = BC$.

Proof. 1. AC is common to the $\triangle s$ ABC and ADC.

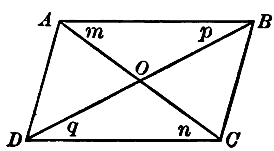
(To be completed.)

202. COROLLARY 1. *Parallel lines included between parallel lines are equal.*

203. COROLLARY 2. *Two parallel lines intercept equal lengths on all lines drawn perpendicular to either of them.*

PROPOSITION XXII. THEOREM

204. *The diagonals of a parallelogram bisect each other.*



Given the parallelogram *ABCD*, with the diagonals *AC* and *BD* intersecting at *O*.

To prove that $AO = OC$, *and* $BO = OD$.

Proof. 1. $\triangle AOB \equiv \triangle COD$. Why?

2. $\therefore AO = OC$, and $BO = OD$.

(To be completed.)

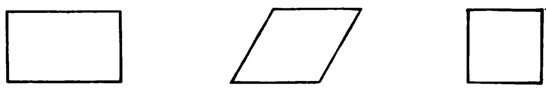
Special Parallelograms

205. A **rectangle** is a parallelogram whose angles are right angles (see § 199).

206. A **rhombus** is an oblique-angled equilateral parallelogram.

NOTE. A parallelogram with oblique angles and with its consecutive sides unequal is sometimes called a **rhomboid.**

207. A **square** is an equilateral rectangle.



RECTANGLE RHOMBUS SQUARE

EXERCISES

1. Prove that the diagonals of a rectangle are equal.

2. Prove that if a parallelogram has equal diagonals, it is a rectangle.

3. Show that a circle can be circumscribed about any rectangle.

4. Prove that the diagonals of a rhombus or a square are perpendicular to each other. Which parallelograms are also kites?

5. Prove that if the diagonals of a parallelogram are perpendicular to each other, the parallelogram is equilateral.

6. Prove that the diagonals of a rhombus bisect the angles through which they pass.

7. Prove that if a diagonal of a parallelogram bisects the angles through which it passes, the parallelogram is equilateral.

8. Make a table of quadrilaterals based upon the relations of their diagonals.

9. How does a carpenter apply Ex. 2 to test whether a window-casing is properly "square-cornered"?

10. How many degrees in each angle of a rhombus if one diagonal is equal to one of the sides?

Proposition XXIII. Theorem

208. *If a quadrilateral has*

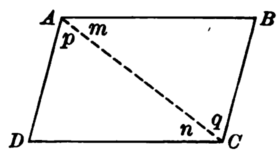
(a) *its opposite sides equal, or*

(b) *one pair of opposite sides equal and parallel, or*

(c) *its diagonals bisecting each other, or*

(d) *its opposite angles equal,*

it is a parallelogram.



(a) **Given the quadrilateral *ABCD*, in which *AB* equals *CD*, and *AD* equals *BC*.**

To prove that $ABCD$ *is a* ▱.

Proof. 1. Draw the diagonal AC.

2. Then	$\triangle ABC \equiv \triangle ADC$.	s. s. s.
For	$AB = CD$, and $AD = BC$,	Hyp.
and	AC is common.	
3.	$\therefore \angle m = \angle n$,	Why?
and hence	$AB \parallel CD$.	Why?
4. Also	$\angle p = \angle q$,	Why?
and hence	$AD \parallel BC$.	Why?
5.	$\therefore ABCD$ is a ▱.	Def.

(b) **Given the quadrilateral *ABCD*, in which *AB* equals *CD*, and *AB* is parallel to *CD*.**

To prove that $ABCD$ *is a* ▱.

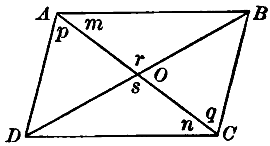
Proof. 1. Draw the diagonal AC.

2. Then	$\triangle ABC \equiv \triangle ADC$.	s. a. s.
3.	$\therefore AD = BC$.	Why?

(To be completed. See (a) above.)

(c) **Given the quadrilateral *ABCD*, and the diagonals *AC* and *BD* bisecting each other at *O*.**

To prove that $ABCD$ *is a* ▱.



Proof. 1. $\triangle AOB \equiv \triangle COD.$ Why?

2. $\therefore \angle m = \angle n,$ Why?

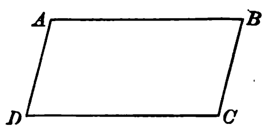
and hence $AB \parallel CD.$ Why?

3. Also $AB = CD.$ Why?

4. $\therefore ABCD$ is a ▱. Why?

(d) **Given the quadrilateral *ABCD*, in which angle *A* equals angle *C*, and angle *B* equals angle *D*.**

To prove that $ABCD$ *is a* ▱.



Proof. 1. $\angle A + \angle B + \angle C + \angle D = 2$ st. $\angle$s. Why?

2. But $\angle A = \angle C,$ and $\angle B = \angle D.$ Hyp.

$\therefore \angle A + \angle B = \angle C + \angle D.$ Ax. 2

That is, $\angle A + \angle B =$ one half the sum of all four $\angle$s.

$\therefore \angle A + \angle B =$ one st. $\angle$.

3. $\therefore AD \parallel BC.$ Why?

4. In like manner it may be shown that

$AB \parallel CD.$

5. $\therefore ABCD$ is a ▱. Why?

EXERCISES

1. Show that each of the cases (a), (b), (c), and (d) of the above theorem affords a method of constructing a parallelogram.

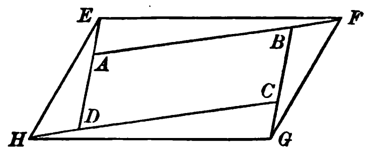
2. How many different parallelograms can be formed by placing together two congruent scalene triangles?

3. How may two congruent isosceles triangles be placed so as to form a rhombus?

4. Given a ▱ *ABCD*, with its sides produced in succession, so that

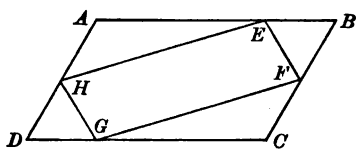
$AE = CG$, and $BF = DH$.

Prove that *EFGH* is a ▱.



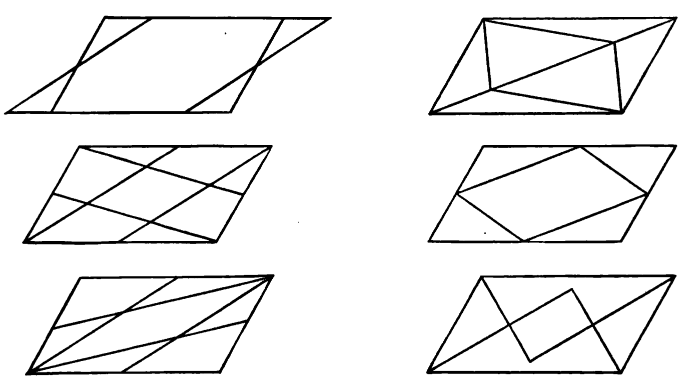
5. If in the above figure *ABCD* were a rectangle (rhombus, square), and $AE = BF = CG = DH$, would *EFGH* be a rectangle (rhombus, square)?

6. Given, in the ▱ *ABCD*, $AH = BE = CF = DG$. Prove that *EFGH* is a ▱.

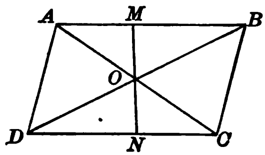


7. If in the above figure *ABCD* were a rectangle (rhombus, square), would *EFGH* be a rectangle (rhombus, square)?

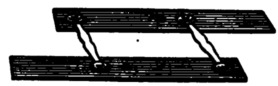
8. What conclusions are suggested by the following diagrams?



9. If $ABCD$ is a ▱, and MN is any line through the intersection of the diagonals in the annexed figure, how many pairs of congruent triangles are formed? Give proof. How many pairs of congruent quadrilaterals? Give proof. Prove that the perimeter is bisected by such a line.

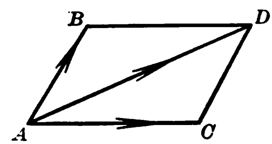


10. If such a ▱ as the above is cut out of cardboard and a pin is thrust through the point O, the cardboard figure will balance in any position. For this reason O is called the center of gravity of the ▱. What reason for this is suggested in the figure?



11. The annexed figure shows the "parallel ruler," which is used by designers for drawing parallel lines. Upon what principle of parallelograms does its construction depend?

12. Parallelogram of Velocities. If a body placed at A has imparted to it at the same time two velocities, represented in magnitude and direction by the lines AB and AC, then the body will acquire an actual velocity represented by AD, the diagonal through A of the ▱ formed on AB and AC as consecutive sides. AD is then called the **resultant velocity,** while AB and AC are called the **component velocities.**



For example, if a boat is rowed across a stream at the rate of 10 mi. an hour, the rate and direction being represented by AC, while the current of the stream is moving at the rate of 5 mi. an hour, its rate and direction being represented by AB, then AD represents the rate and the direction in which the boat is actually moving.

Construct with ruler and protractor, and measure the missing elements in the following table (referred to the above figure):

AB	AC	$\angle BAC$	AD	$\angle DAC$
5	12	90°	?	?
13	14	60°	?	?
8	15	45°	?	?
17	20	60°	?	?

209. The scheme of § 167 now becomes as follows:

In order to prove two (lines / angles) equal:

I. Show that they are homologous (sides / angles) of congruent ∆s.

In order to prove two triangles congruent:

1. Show that they have the relation a. s. a.
2. Show that they have the relation s. a. s.
3. Show that they have the relation s. s. s.
4. Show that they have the relation rt. △ h. l.
5. Show that they have the relation rt. △ h. a.

II. Show that in a △ they are (sides opposite equal ∠s / ∠s opposite equal sides).

III. Show that they are (∥ lines included between / alt.-int. ∠s, corr. ∠s, alt.-ext. ∠s of) ∥ lines.

In order to prove that two lines are ∥:

1. Show that a transversal makes alt.-int. ∠s equal.
2. Show that a transversal makes ext.-int. ∠s equal.
3. Show that a transversal makes alt.-ext. ∠s equal.
4. Show that a transversal makes int. ∠s on the same side supplementary.
5. Show that a transversal makes ext. ∠s on the same side supplementary.
6. Show that they are ⊥ to the same line.
7. Show that they are ∥ to the same line.
8. Show that they are opposite sides of a ▱.

IV. Show that they are opposite (sides / ∠s) of a ▱.

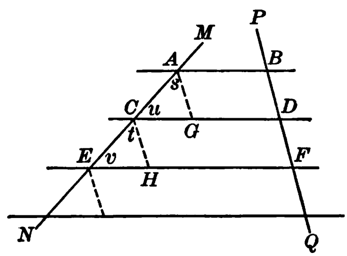
In order to prove that a quadrilateral is a ▱:

1. Show that its opposite sides are parallel.
2. Show that its opposite sides are equal.
3. Show that one pair of opposite sides are equal and parallel.
4. Show that its diagonals bisect each other.
5. Show that its opposite angles are equal.

THE TRANSVERSAL THEOREM. TRAPEZOIDS

PROPOSITION XXIV. THEOREM

210. *If three or more parallels intercept equal parts on one transversal, they intercept equal parts on every transversal.*



Given the parallels *AB*, *CD*, and *EF* intercepting equal parts on the transversal *MN*.

***To prove** that they intercept equal parts on any other transversal, as PQ.*

Proof. (Outline).

1.	Draw AG and CH both $\parallel$ to PQ.	
2. Then	$AG \parallel CH$.	§ 175
3. Now	$\triangle ACG \equiv \triangle CEH$.	Why?
4.	$\therefore AG = CH$.	
5. But	AD is a $\square$, and $AG = BD$.	Why?
6. Also	CF is a $\square$, and $CH = DF$.	Why?
7.	$\therefore BD = DF$.	Why?

In like manner the other segments on PQ are proved equal.

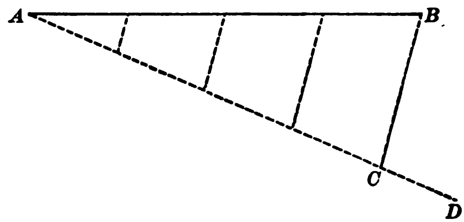
That is, the parallels intercept equal parts on PQ.

211. COROLLARY. *A line bisecting one side of a triangle, and parallel to another side, bisects the third side.*

The third parallel required by the above theorem is constructed through the vertex of the given triangle.

Proposition XXV. Problem

212. *To divide a given straight line into any number of equal parts.*



Given the straight line *AB*.

***Required** to divide AB into any number of equal parts.*

Construction. 1. From A draw the line AD, making any convenient angle with AB.

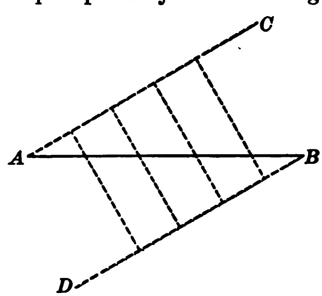
2. On AD lay off any convenient length a number of times equal to the number of parts into which AB is to be divided.

3. From C, the last point thus found on AD, draw CB.

4. Through the other points of division on AD draw lines parallel to CB. These lines will divide AB into equal parts. Why?

EXERCISES

1. A line AB may be divided into equal parts by the following method, which can be effected more rapidly than the one preceding: In the annexed figure draw $AC \parallel$ to BD. Lay off on these two lines equal segments, the same number on each. Join the corresponding points as shown. Prove the construction.



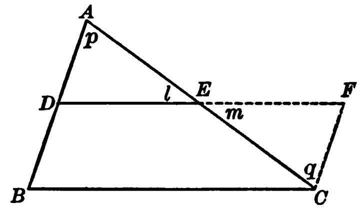
2. A sheet of ruled paper is sometimes of use in dividing a given line into equal parts. Explain.

3. Divide a given line into 3 (4, 5, 6, 7) equal parts, using a sheet of ruled paper.

4. Divide a given line in the ratio of 2 : 3 (3 : 5, 1 : 4, 1 : 2 : 3).

Proposition XXVI. Theorem

213. *The line which joins the mid-points of two sides of a triangle is parallel to the third side and equal to half the third side.*



Given the triangle *ABC*, and the line *DE* bisecting *AB* and *AC*.

To prove that $DE \parallel BC$, *and that* $DE = \frac{1}{2} BC$.

Proof. 1. Produce DE, and draw CF parallel to AB, meeting DE produced in F.

2. Then	$\triangle AED \equiv \triangle ECF$.	a. s. a.
For	$AE = EC$,	Hyp.
	$\angle l = \angle m$,	Why ?
and	$\angle p = \angle q$.	Why ?
3.	$\therefore AD = FC$.	Why ?
But	$AD = DB$.	Hyp.
	$\therefore FC = DB$.	Ax. 1
4. Now	$FC \parallel DB$.	Cons.
	$\therefore DFCB$ is a ▱.	Why ?
5.	$\therefore DE \parallel BC$, and $DF = BC$.	Why ?
6. But	$DE = EF$.	Why ?
That is,	$DE = \frac{1}{2} DF = \frac{1}{2} BC$.	

214. Proposition XXVI illustrates the following principle:

In order to prove that one line is half of another line,

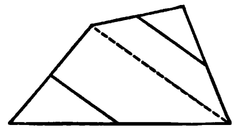
(1) **double the smaller line, or**

(2) **bisect the larger line,**

and proceed according to § 209.

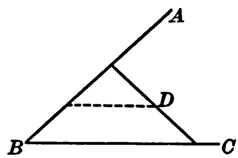
EXERCISES

1. The line which joins the mid-points of two adjacent sides of any quadrilateral is equal and parallel to the line which joins the mid-points of the other two sides.



2. What kind of quadrilateral is formed by lines which join in succession the mid-points of the sides of any quadrilateral? of a kite? of a rectangle? of a rhombus? of a square? Prove your answer in each case.

3. The lines which join the mid-points of the sides of a triangle divide it into four congruent triangles.



4. PROBLEM. Given the angle ABC and the point D within it, to draw a line through D terminating in the sides of the angle and bisected at D.

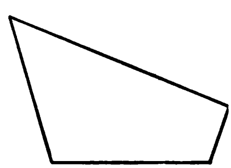
215. A **trapezoid** is a quadrilateral two and only two of whose sides are parallel. The nonparallel sides of a trapezoid are called the **legs,** and the parallel sides the **bases.**

The cross section of a canal bank, or of a trench, or of a breakwater is often a trapezoid. Give other common examples of the trapezoid.

216. The line joining the mid-points of the legs is called the **mid-line** of the trapezoid.

217. An **isosceles trapezoid** is one whose legs are equal.

The isosceles trapezoid often arises in geometry from the cutting off of part of an isosceles triangle by a line parallel to the base.

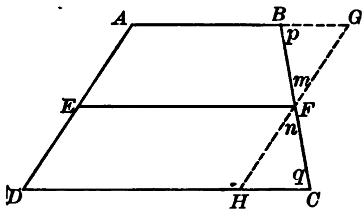


218. A **trapezium** is a quadrilateral which has no two sides parallel.

Have examples of the trapezium already appeared in this text? Where?

Proposition XXVII. Theorem

219. *The mid-line of a trapezoid is parallel to the bases and equal to half their sum.*



Given the trapezoid *ABCD*, and the line *EF* bisecting the legs *AD* and *BC* in *E* and *F* respectively.

To prove that $EF \parallel AB$ and CD, and that $EF = \frac{1}{2}(AB + CD)$.

Proof. 1. Through F draw GH parallel to AD, cutting DC in H, and AB produced in G.

2. Then	$\triangle BFG \equiv \triangle HFC$.	a. s. a.
	$\therefore FG = FH$.	Why?
3. But	$AGHD$ is a $\square$.	Why?
	$\therefore AD = GH$.	Why?
Whence	$AE = FG$.	Ax. 5
4.	$\therefore AGFE$ is a $\square$.	Why?
5.	In like manner $EFHD$ is a $\square$.	
6.	$\therefore EF \parallel AG$, and $EF = AG$.	Why?
Also	$EF \parallel DH$, and $EF = DH$.	

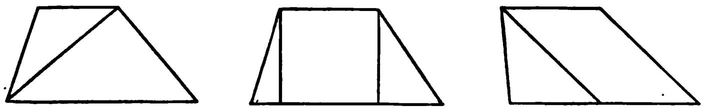
7. $\therefore EF = \frac{1}{2}(AG + DH) = \frac{1}{2}(AB + BG + DH)$

$= \frac{1}{2}(AB + HC + DH)$ Why?

$= \frac{1}{2}(AB + CD)$.

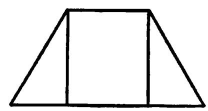
220. Corollary. *A line which bisects one leg of a trapezoid and is parallel to the bases bisects the other leg also.*

EXERCISES



1. From the annexed figures it appears that:

(a) Any trapezoid may be broken up into two triangles (by a diagonal), or into a rectangle and two right triangles, or into a triangle and a parallelogram. Give proof.



(b) An isosceles trapezoid may be broken up into a rectangle and two congruent right triangles. Why?

2. Construct a trapezoid, having given the four sides.

3. Construct a trapezoid, having given the two bases and the two base angles.

4. Prove that each base of an isosceles trapezoid makes equal angles with the legs.

5. Prove that if a trapezoid has equal base angles, it is isosceles.

6. Prove that the diagonals of an isosceles trapezoid are equal.

7. Prove that if a trapezoid has equal diagonals, it is isosceles.

Suggestion. Draw ⊥s from the ends of the upper base upon the lower.

INEQUALITIES

221. It now becomes necessary to prove, not that two magnitudes are equal, but that one of two magnitudes is greater than the other.

The signs of inequality, $>$ (greater than) and $<$ (less than), are said to indicate the *sense* of the inequality.

222. Axioms of Inequality. Proofs involving inequalities presuppose the following axioms (see § 132).

8. *The whole of any magnitude is greater than any part of it.*

9. *If equals are added to unequals, the results are unequal in the same sense.*

Thus, if $a > b$, and $c = d$, then $a + c > b + d$.

10. *If equals are subtracted from unequals, the remainders are unequal in the same sense.*

11. *If unequals are subtracted from equals, the results are unequal in the opposite sense.*

That is, if $a = b$ and $c > d$, then $a - c < b - d$.

12. *If of three magnitudes the first is greater than the second, and the second is greater than the third, then the first is greater than the third.*

That is, if $a > b$ and $b > c$, then $a > c$.

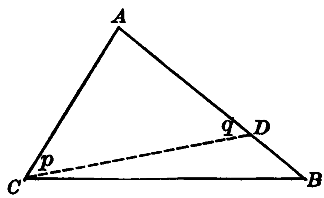
223. Two inequalities have already been established:

The exterior angle of a triangle is greater than either remote interior angle.

In a right or an obtuse triangle the right or the obtuse angle is the greatest angle.

PROPOSITION XXVIII. THEOREM

224. *If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.*



Given the triangle *ABC*, in which *AB* is greater than *AC*.

To prove *that* $\angle ACB > \angle B$.

Proof. 1. On AB lay off $AD = AC$.

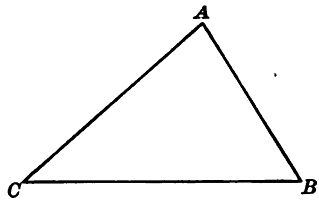
(This is possible since $AB > AC$, by hyp.)

Draw CD.

2. Then	$\angle p = \angle q$.	Why?
3. But	$\angle ACB > \angle p$,	Ax. 8
and	$\angle q > \angle B$.	§ 223
4.	$\therefore \angle ACB > \angle B$.	Axs. 1 and 12

Proposition XXIX. Theorem

225. *If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle.*



Given the triangle ABC, in which angle B is greater than angle C.

To prove *that* $AC > AB$.

Proof. 1. Now $AC = AB$, or $AC < AB$, or $AC > AB$.

2. If $AC = AB$, then $\angle B = \angle C$. Why?

But this is contrary to the hypothesis.

3. If $AC < AB$, then $\angle B < \angle C$. § 224

But this also is contrary to the hypothesis.

4. $\therefore AC > AB$.

(This proposition is the converse of Proposition XXVIII.)

226. Corollary 1. *In a right triangle the hypotenuse is longer than either of the legs.*

227. Corollary 2. *Of all the lines that can be drawn from a given point to a given line, the perpendicular is the shortest.*

EXERCISES

1. The angles at the extremities of the greatest side of a triangle are acute.

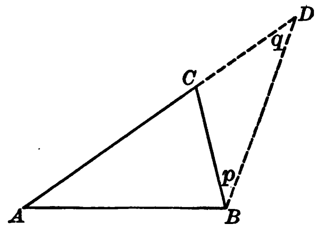
2. If an isosceles triangle is obtuse, the base is the longest side.

3. If the diagonals of a parallelogram are unequal, the angles of the parallelogram must be oblique.

4. In a $\square ABCD$ the side $AB > CB$. Draw the diagonal AC and prove that $\angle ACB > \angle ACD$.

Proposition XXX. Theorem

228. *In any triangle the sum of two sides is greater than the third side.*



Given the triangle ABC.

To prove that $AC + CB > AB$.

Proof. 1. Produce AC to D, making $CD = CB$. Draw BD.

2. Then $\angle p = \angle q$. Why?

3. But $\angle ABD > \angle p$. Why?

$\therefore \angle ABD > \angle q$.

4. $\therefore AD > AB$. § 225

That is, $AC + CB > AB$.

(Since $CD = CB$.)

229. Corollary 1. *In any triangle any side is greater than the difference of the other two.*

Since $AC + CB > AB$,

$\therefore AC > AB - CB$. Ax. 10

230. Corollary 2. *In any polygon any side is less than the sum of the other sides.*

Suggestion. Prove by drawing all the diagonals from one end of the side taken, and applying § 228 and Ax. 12 to the successive triangles thus formed.

Remark. Proposition XXX is often given among the self-evident truths of geometry, and it may be so classified. The proof given above is due to Euclid (300 B.C.).

EXERCISES

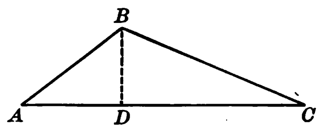
1. Given a $\triangle ABC$, with $AB > AC$. The bisectors of $\angle B$ and $\angle C$ meet at P. Prove that $BP > CP$.

2. Given a $\triangle ABC$, with $AB > AC$. The altitudes from B and C meet at O. Prove that $OB > OC$.

3. The median to any side of a triangle is less than half the sum of the other two sides.

Suggestion. Extend the median its own length.

4. The straight line joining the vertex of an isosceles triangle to any point in the base produced is greater than either of the equal sides.

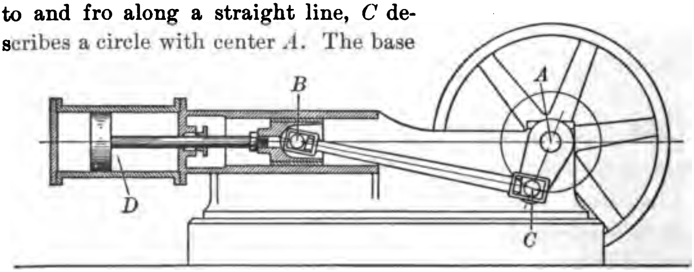


5. Prove Proposition XXX by using the annexed diagram, in which $BD \perp AC$.

6. Can a triangular frame be made by hinging together rods whose lengths are 8 in., 5 in., 3 in.? Explain.

7. Given four rods of lengths 3 in., 6 in., 7 in., 10 in. How many different triangular frames could be made by hinging any three of these rods together at their extremities?

8. The diagram illustrates the mechanism of the connecting rod of a steam engine. BC represents the connecting rod, CA the crank, BD the piston rod. While the piston rod BD causes B to move to and fro along a straight line, C describes a circle with center A. The base



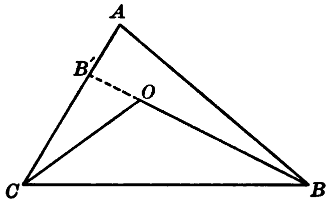
AB of $\triangle ABC$ constantly changes. The extreme positions of B are called the "dead points." If BC is 4 ft. long and AC 10 in. long, within what values does AB vary?

9. The straight line joining the vertex of an isosceles triangle to any point in the base is less than either of the equal sides of the triangle.

10. The perimeter of a quadrilateral is greater than the sum of its diagonals.

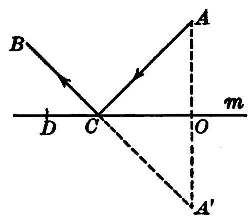
11. If O is any point within a $\triangle ABC$, prove that $OB + OC < AB + AC$.

(Extend OB to meet AC.)



12. A ray of light starting at A is reflected by a plane mirror m to the point B (see p. 107, Ex. 32). Locate the point where the ray strikes the mirror.

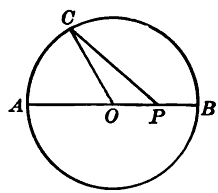
(Draw $OA \perp$ to m, make $OA' = OA$, etc.)



13. Prove that the path of the ray in Ex. 12, that is, $AC + CB$, is shorter than $AD + DB$, D being any other point on m.

14. The diameter of a circle is greater than any other chord.

(Join the extremities of the chord to the center.)



15. In the figure AB is a diameter. Prove that $PA > PC$, and that $PC > PB$.

(Draw OC.)

16. The altitude on any side of a triangle is less than half the sum of the other two sides.

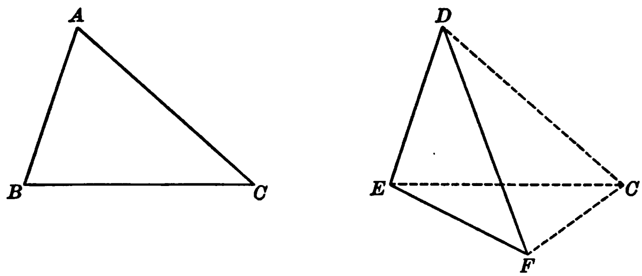
17. The sum of three altitudes of a triangle is less than the perimeter of the triangle.

18. Prove Proposition XXVIII by drawing the bisector of the $\angle A$ to meet the side CB in E, and joining E to a point D on AB such that AD is equal to AC.

19. Two regular polygons (§ 57), a square and an octagon, are constructed about the same center (§ 95) with equal radii. Prove that the perimeter of the octagon is greater than the perimeter of the square.

PROPOSITION XXXI. THEOREM

231. *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



Given the triangles *ABC* and *DEF*, in which *AB* equals *DE*, *AC* equals *DF*, and the angle *A* is greater than the angle *D*.

To prove that $BC > EF$.

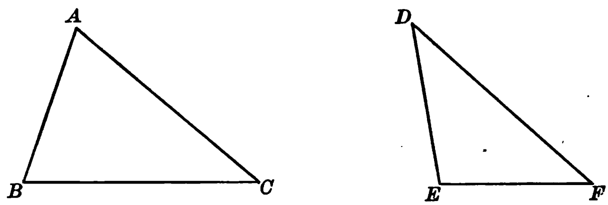
Proof. 1. Construct $\triangle DEC$ congruent to $\triangle ABC$, as shown in the figure, and draw CF.

Then	DC falls without $\angle EDF$.	Why?
2. Now	$DC = DF$.	Hyp.
	$\therefore \angle DFC = \angle DCF$.	Why?
3. But	$\angle EFC > \angle DFC$.	Why?
	$\therefore \angle EFC > \angle DCF$.	Ax. 1
4. Also	$\angle DCF > \angle ECF$.	Why?
	$\therefore \angle EFC > \angle ECF$.	Ax. 12
5.	$\therefore EC > EF$.	§ 225
That is,	$BC > EF$.	

Discussion. If F falls on EC, the proposition is self-evident. If F falls within the triangle DEC, the proof is similar to the one given above.

Proposition XXXII. Theorem

232. *If two triangles have two sides of the one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*



Given the triangles *ABC* and *DEF*, in which *AB* equals *DE*, *AC* equals *DF*, and *BC* is greater than *EF*.

***To prove** that* $\angle A > \angle D$.

Proof. 1. Now $\angle A = \angle D$, or $\angle A < \angle D$, or $\angle A > \angle D$.

2. If $\angle A = \angle D$, then $BC = EF$. Why?

But this is contrary to the hypothesis.

3. If $\angle A < \angle D$, then $BC < EF$. § 231

But this also is contrary to the hypothesis.

4. $\therefore \angle A > \angle D$.

(This proposition is the converse of Proposition XXXI.)

EXERCISES

1. In a triangle ABC, $AC > AB$. Equal distances BD and CE are laid off on BA and CA respectively. Prove that $CD > BE$.

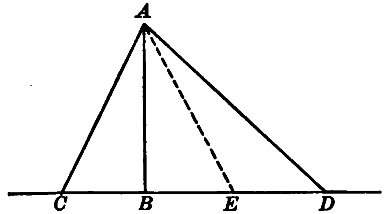
2. In the $\triangle ABC$, D is the mid-point of the side BC, and $AB > AC$. Prove that $\angle ADC$ is acute.

3. In the acute $\triangle ABC$, $AB > AC$. If AD is the median, and AE the altitude to the side BC, prove that E lies between C and D.

4. If any point P within the $\triangle ABC$ is joined to B and C, prove that $\angle BPC > \angle BAC$.

Proposition XXXIII. Theorem

233. *Of two straight lines drawn from the same point in a perpendicular to a given line and cutting off on the line unequal segments from the foot of the perpendicular, the more remote is the greater.*



Given the line *CD*, *AB* perpendicular to *CD*, and *BD* greater than *BC*.

***To prove* that** $AD > AC$.

Proof. 1. Take $BE = BC$, and draw AE.

2. Then	AE falls within the $\angle BAD$,	Why?
and	$\triangle ABC \equiv \triangle ABE$.	s. a. s.
3.	$\therefore AE = AC$.	Why?
4. Now	$\angle AEB$ is an acute $\angle$.	Why?
	$\therefore \angle AED$ is an obtuse $\angle$.	Why?
Also	$\angle ADB$ is an acute $\angle$.	Why?
5.	$\therefore AD > AE$.	§ 225
That is,	$AD > AC$.	Ax. 1

234. Corollary. *Only two equal straight lines can be drawn from a point to a straight line; and of two unequal lines from a point to a line, the greater cuts off the greater segment from the foot of the perpendicular.*

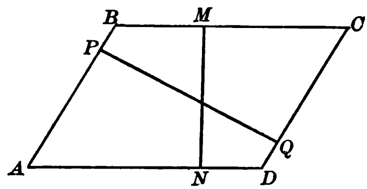
Prove from Proposition XXXIII by the Indirect Method.

235. The **distance between two points** is the length of the line-segment joining them. § 228

236. The **distance from a point to a line** is the length of the perpendicular let fall from that point upon the line. § 227

237. The **distance between two parallel lines** is the length of the segment of a common perpendicular cut off between the two lines.

238. The **altitude of a parallelogram or of a trapezoid** is the perpendicular distance between the bases, as MN. (AD and BC are called the lower and the upper base respectively.)



PQ is also an altitude of the $\square ABCD$, the corresponding bases being AB and CD.

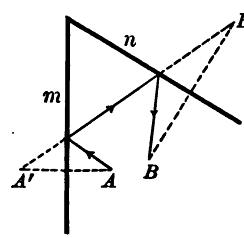
EXERCISES

1. In a quadrilateral $DEFH$, $DH = EF$, and $\angle H > \angle F$. Prove that $DF > EH$.

2. In a quadrilateral $DEFH$, $DH = EF$, and $DF > EH$. Prove that $\angle H > \angle F$.

3. If P is any point within the $\triangle ABC$, prove that $PA + PB + PC$ is less than the perimeter, but greater than half the perimeter, of the triangle (see Ex. 11, p. 129).

4. If P is any point within a quadrilateral, prove that the sum of its distances from the vertices is not less than the sum of the diagonals. When is it equal to the sum of the diagonals?



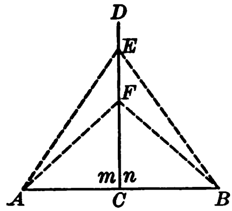
5. The sum of the distances of any point within a polygon from the vertices is greater than half the perimeter of the polygon.

6. Two plane mirrors m and n meet at an angle. The points A and B are within the angle mn. The figure shows the path of a ray of light emanating from A and reflected by the mirrors m and n through B. Take any other two points, one each in m and n, join them by a line-segment, and draw the lines from the point in m to A and from the point in n to B. Show that the length of this path from A to B is greater than the path of the ray of light. This exercise illustrates the fact that light travels by the *shortest* path.

COLLINEARITY AND CONCURRENCE

PROPOSITION XXXIV. THEOREM

239. *All points in the perpendicular bisector of a line are equidistant from the extremities of the line; and conversely, all points equidistant from the extremities of the line lie in the perpendicular bisector.*



(a) **Given *DC*, the perpendicular bisector of *AB*.**

To prove *that all points in DC are equidistant from A and B.*

Proof. 1. Take any point in DC, as E. Draw AE and BE.

2. Then $\triangle AEC \equiv \triangle BEC$. Why?

3. $\therefore AE = BE$. Why?

4. $\therefore$ since E is *any* point in DC, *all* points in DC are equidistant from A and B.

(b) **Given the line *AB*.**

To prove *that all points equidistant from A and B lie in the perpendicular bisector of AB.*

Proof. 1. Take any such equidistant point as F. Draw FA and FB. Then $FA = FB$. Draw FC bisecting AB.

2. Then $\triangle AFC \equiv \triangle BFC$. Why?

3. $\therefore \angle m = \angle n$. Why?

$\therefore FC \perp AB$. Why?

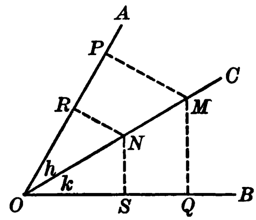
4. That is, the point F lies in the perpendicular bisector of AB.

5. But F is *any* point equidistant from A and B.

$\therefore$ *all* such points lie in the perpendicular bisector of AB.

PROPOSITION XXXV. THEOREM

240. *All points in the bisector of an angle are equidistant from the sides of the angle; and conversely, all points equidistant from the sides of an angle lie in the bisector of the angle.*



(a) **Given the angle AOB bisected by OC.**

To prove *that all points in OC are equidistant from OA and OB.*

Proof. 1. Take any point in OC, as M. Draw $MP \perp$ to OA, and $MQ \perp$ to OB.

2. Then $\triangle MOP \equiv \triangle MOQ$. Why?

3. $\therefore MP = MQ$. Why?

4. $\therefore$ since M is *any* point in OC, *all* points in OC are equidistant from OA and OB.

(b) **Given the angle AOB.**

To prove *that all points equidistant from OA and OB lie in the bisector of the $\angle AOB$.*

Proof. 1. Take any such equidistant point as N. Draw $NR \perp$ to OA, and $NS \perp$ to OB. Then $NR = NS$. Draw NO.

2. Then $\triangle NOR \equiv \triangle NOS$. Why?

3. $\therefore \angle h = \angle k$, that is, NO bisects $\angle AOB$.

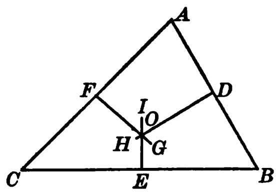
4. That is, the point N lies in the bisector of the $\angle AOB$.

5. But N is *any* point equidistant from the sides of the $\angle AOB$.

$\therefore$ *all* points equidistant from OA and OB lie in the bisector of the $\angle AOB$.

Proposition XXXVI. Theorem

241. *The perpendicular bisectors of the sides of a triangle meet in a point equidistant from the three vertices.*



Given the triangle *ABC*, with *DH*, *EI*, and *FG* the perpendicular bisectors of the sides *AB*, *BC*, and *CA* respectively.

To prove *that DH, EI, and FG meet in a point equidistant from A, B, and C.*

Proof. 1.	*DH* and *FG* will intersect.	§ 178
	(Since they cannot be ∥, being ⊥ to two intersecting lines.)	
	Call the point of intersection *O*.	
2. Then	*O* is equidistant from *A* and *B*.	§ 239 (a)
Also	*O* is equidistant from *A* and *C*.	§ 239 (a)
3.	∴ *O* is equidistant from *B* and *C*.	Ax. 1
	∴ *O* lies in *EI*.	§ 239 (b)

That is, *DH*, *EI*, and *FG* meet in *O*, which is equidistant from *A*, *B*, and *C*.

242. The point *O* is called the **circumcenter** of the △*ABC*.

243. Three or more lines which meet in a point are said to be **concurrent**.

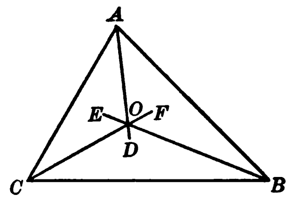
Thus, from the above proposition, the perpendicular bisectors of the sides of a triangle are concurrent.

244. Three or more points which lie in the same line are said to be **collinear**.

For example, from § 239, points equidistant from the extremities of a line are collinear.

PROPOSITION XXXVII. THEOREM

245. *The bisectors of the angles of a triangle are concurrent in a point equidistant from the three sides.*



Given the triangle *ABC*, with *AD*, *BE*, and *CF* the bisectors of the angles *A*, *B*, and *C* respectively.

To prove *that AD, BE, and CF are concurrent in a point equidistant from the sides of the triangle.*

Proof. 1. *AD* and *BE* will intersect.

(Since they cannot be ‖, the sum of the ∠s which they make with the transversal *AB* being less than two rt. ∠s.)

Call the point of intersection *O*.

2. Then	*O* is equidistant from *AC* and *AB*.	§ 240 (a)
Also	*O* is equidistant from *AB* and *BC*.	§ 240 (a)
3.	∴ *O* is equidistant from *AC* and *BC*.	Ax. 1
	∴ *O* lies in *CF*.	§ 240 (b)

That is, *AD*, *BE*, and *CF* are concurrent in *O*, which is equidistant from the three sides.

246. The point *O* is called the **incenter** of the △*ABC*.

EXERCISES

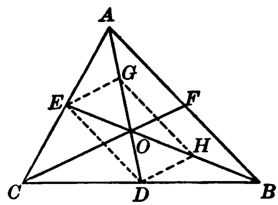
1. What construction is suggested by Proposition XXXVI?

2. Does the incenter of a triangle always fall within the triangle?

3. In the △*ABC* show that the bisectors of the interior angle *A* and the exterior angles at *B* and *C* are concurrent in a point equidistant from the sides of the triangle. This point is called an **excenter** of the △*ABC*. How many excenters has a triangle?

PROPOSITION XXXVIII. THEOREM

247. *The medians of a triangle are concurrent in a point which is two thirds of the distance from each vertex to the middle of the opposite side.*



Given the triangle *ABC*, with the medians *AD*, *BE*, and *CF* to the sides *BC*, *CA*, and *AB* respectively.

To prove *that AD, BE, and CF are concurrent in a point two thirds of the distance from each vertex to the middle of the opposite side.*

Proof. 1. AD and BE will intersect.

(Since one of them joins two points which lie on opposite sides of the other.)

Name the point of intersection O. Bisect AO and BO in G and H respectively. Draw ED and GH, also EG and DH.

2. Then	$DE \parallel AB$, and $DE = \frac{1}{2} AB$.	Why?
Also	$GH \parallel AB$, and $GH = \frac{1}{2} AB$.	Why?
	$\therefore DE = GH$ (Ax. 1), and $DE \parallel GH$.	Why?
	$\therefore EGHD$ is a $\square$.	Why?
Whence	$EO = OH = \frac{1}{2} OB$.	Why?
3.	$\therefore BO = 2\,OE = \frac{2}{3} BE$.	
Also	$AO = \frac{2}{3} AD$.	

That is, AD and BE intersect in a point which is two thirds of the distance from each vertex to the middle of the opposite side.

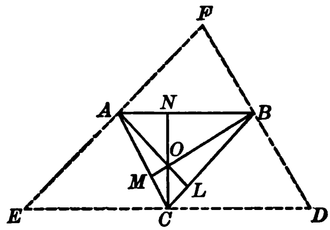
4. In like manner it can be proved that CF and AD intersect in a point which is two thirds of the distance from each vertex to the middle of the opposite side. That is, CF passes through O.

$\therefore AD$, BE, and CF are concurrent in a point two thirds of the distance from each vertex to the middle of the opposite side.

248. The point O is called the **centroid** of the $\triangle ABC$.

Proposition XXXIX. Theorem

249. *The altitudes of a triangle are concurrent.*



Given the triangle *ABC*, with the altitudes *AL*, *BM*, *CN* respectively.

To prove that *AL, BM, and CN are concurrent.*

Proof. 1. Through A, B, and C respectively draw EF, FD, DE $\parallel$ to BC, CA, and AB respectively.

2.	Then $FBCA$ is a ▱.	Why?
	$\therefore FB = AC$.	Why?
Also	$BDCA$ is a ▱.	Why?
	$\therefore BD = AC$.	Why?
3.	$\therefore FB = BD$.	Ax. 1
4. Now	$BM \perp AC$.	Hyp.
	$\therefore BM \perp FD$.	Why?

That is, *BM* is the perpendicular bisector of *FD*.

(To be completed.)

250. The point O is called the **orthocenter** of the $\triangle ABC$.

EXERCISES

1. Cut a triangle out of cardboard. Find its centroid and thrust a pin through this point. The triangle will balance in whatever vertical position it is placed. This shows that the centroid is the **center of gravity** of the triangle. A reason why this should be so will be given in Book IV.

2. In an equilateral triangle the incenter, the circumcenter, the orthocenter, and the centroid are the same point.

MISCELLANEOUS EXERCISES

1. The perpendiculars upon the legs of an isosceles triangle from the mid-point of the base are equal.

2. The sum of the perpendiculars from a point in the base of an isosceles triangle to the legs is equal to the altitude upon one of the legs (Fig. 1).

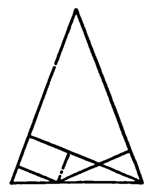


FIG. 1

3. The sum of the perpendiculars from any point within an equilateral triangle to the three sides is equal to the altitude of the triangle (Fig. 2).

4. If two medians of a triangle are equal, the triangle is isosceles.

5. The perpendiculars from the mid-points of two sides of a triangle upon the third side are equal.

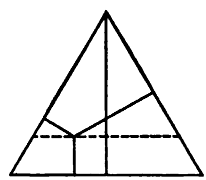


FIG. 2

6. The lines drawn through two opposite vertices of a parallelogram to the mid-points of the opposite sides divide one of the diagonals into three equal parts (Fig. 3).

7. If upon the three sides of a triangle equilateral triangles are constructed lying outside the given triangle, the lines joining the outer vertices of the equilateral triangles to the opposite vertices of the given triangle are equal.

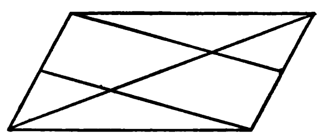


FIG. 3

8. If on one leg of an isosceles triangle, and on the other leg produced, equal lengths are laid off on opposite sides of the base, the line joining the points thus determined is bisected by the base (Fig. 4).

9. The line joining the mid-points of the diagonals of a trapezoid is equal to half the difference of the bases.

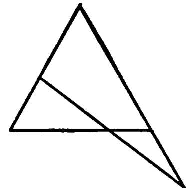


FIG. 4

10. Two triangles are congruent if they have a side and the altitude and the median on that side in one equal to corresponding parts in the other.

11. Composite Diagrams. Examine each of the following figures with reference to (a) equal lines, (b) equal angles, (c) congruent triangles, (d) parallel lines, (e) parallelograms, (f) particular lines (bisectors, medians, altitudes, etc.), (g) symmetrical figures, (h) regular figures.

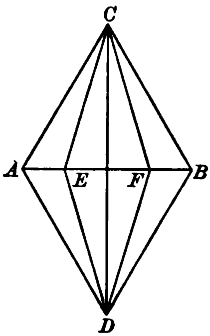


FIG. 5

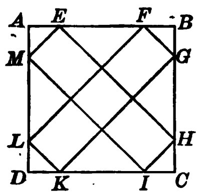


FIG. 6

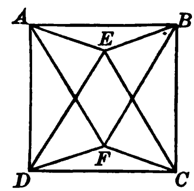


FIG. 7

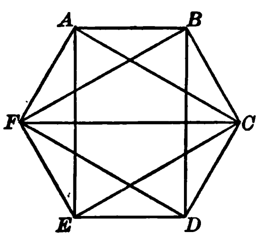


FIG. 8

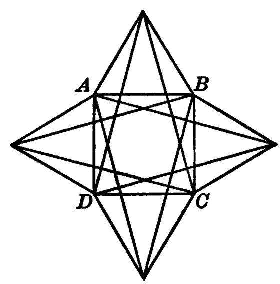


FIG. 9

Fig. 5. *ABC* and *ABD* are congruent isosceles triangles. *AB* is divided into four equal parts.

Fig. 6. *ABCD* is a square.

$$AE = AM = BF = BG = CH = CI = DK = DL.$$

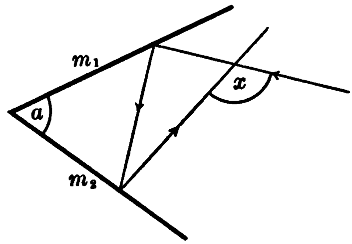
Fig. 7. *ABCD* is a square. *ABF* and *CDE* are equilateral triangles.

Fig. 8. *ABCDEF* is a regular hexagon.

Fig. 9. *ABCD* is a square, with equilateral triangles on its sides.

12. The bisectors of two exterior angles of a triangle meet at an angle which is equal to one half the third exterior angle.

13. Two mirrors, m_1 and m_2, are set so as to form an acute angle with each other. A ray of light is reflected by m_1 so as to strike m_2. The ray is again reflected by m_2 and crosses its first path. Prove that $\angle x = 2\angle a$. (This is the principle underlying several important optical instruments, such as the **sextant**.)



14. The circumcenter of a right triangle is the mid-point of the hypotenuse; of an obtuse triangle, a point without the triangle.

15. If the bisectors of the angles of a quadrilateral are concurrent, the sum of two opposite sides equals the sum of the other two opposite sides.

16. Two trapezoids are congruent if their sides are respectively equal.

17. The lines which join in succession the mid-points of the sides of an isosceles trapezoid inclose a rhombus.

18. If the bisectors of two opposite angles of a quadrilateral form a straight line, the bisectors of the other two angles meet on that line.

19. If the bisectors of the interior angles of a quadrilateral are not concurrent, they inclose a quadrilateral of which the opposite angles are supplementary.

20. Prepare a summary of the most important topics in Book I.

21. Give a list of construction lines that have been helpful in Book I.

22. What methods of proving lines or angles equal have been given so far?

23. Prepare a list of all propositions which, directly or indirectly, justify Proposition XXXVIII. (These may be arranged by number, according to their relative dependence, in the form of a pyramid.)

24. Prepare a summary of the applied problems given in the exercises of Book I.

BOOK II

THE CIRCLE

251. Preliminary Propositions. The following properties of circles have been either proved or taken for granted previously:

1. *A circle can be drawn about any given point as a center, with a radius equal to any given line.*

2. *Radii of the same circle are equal* (§ 60).

3. *Circles which have equal radii are equal* (§ 61).

4. *A point is within, on, or without a circle, according as the line which joins the point to the center is less than, equal to, or greater than the radius of the circle* (§ 62).

5. *A secant to a circle intersects the circle in two and only two points* (§ 68).

6. *If the center segment of two circles is less than the sum but greater than the difference of their radii, the circles intersect in two and only two points* (§ 71).

7. *In the same circle or in equal circles:*
   Equal central angles intercept equal arcs (§ 85).
   Equal arcs are intercepted by equal central angles (§ 85).
   Equal chords subtend equal arcs (§ 86).
   Equal arcs are subtended by equal chords (§ 86).

8. *A central angle is measured by its intercepted arc* (§ 94).

9. *In the same circle, or in equal circles, the greater central angle intercepts the greater arc, and the greater arc is intercepted by the greater central angle* (§ 94).

10. *A diameter of a circle bisects the circle* (§ 63).

NOTE. A circle may be conveniently named by its center. Thus a circle whose center is O will hereafter be called "the circle O." The word "circle" is often replaced by the symbol ⊙.

EXERCISES

1. Two points P and Q are 2 cm. apart. Draw a circle, of radius 3 cm., passing through P and Q.

2. The centers of two circles are 4 cm. apart. The radii of the circles are 3 cm. and 5 cm. respectively. Describe the relative position of the two circles.

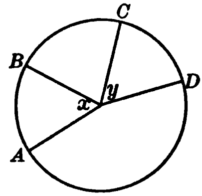


FIG. 1

3. If, in Fig. 1, $\angle x = \angle y$, prove that arc ABC = arc BCD.

4. If, in Fig. 2, chord AB = chord BC = chord CD, prove that chord AC = chord BD.

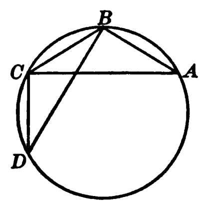


FIG. 2

5. In the same figure, prove that $\angle A = \angle D$. What other angles are equal?

6. If the sides of an inscribed polygon are equal, prove that the polygon is equiangular (see Ex. 5).

7. In Fig. 3, $BC = AD$. Prove that $AB = CD$.

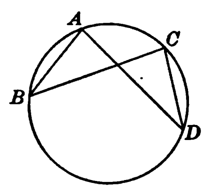


FIG. 3

8. If two equal chords of a circle intersect, the segments of one are equal respectively to the segments of the other.

9. In Fig. 4, $AB = CD$. Prove that $\triangle ABD \equiv \triangle CDB$, and that $\triangle AOD \equiv \triangle COB$.

10. In the same figure prove that $\triangle BOD$ is isosceles.

11. If from a point O on a circle (Fig. 5) a chord OA and a diameter OB are drawn, prove that a radius parallel to OA bisects the arc AB. (Draw PA.)

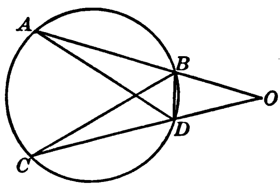


FIG. 4

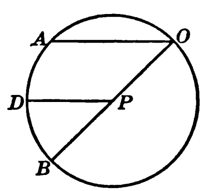
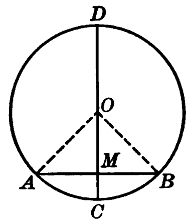


FIG. 5

CHORDS AND SECANTS

PROPOSITION I. THEOREM

252. *A diameter perpendicular to a chord bisects the chord and the arcs subtended by it.*



Given the circle *O*, with a diameter *DC* perpendicular to the chord *AB* at *M*, cutting the minor arc *ACB* at *C*, and the major arc *ADB* at *D*.

To prove *that $AM = BM$, arc AC = arc CB, and arc AD = arc DB.*

Proof. 1. Draw the radii OA and OB.

2. Then rt. $\triangle AOM \equiv$ rt. $\triangle BOM$. Why?

3. $\therefore AM = BM$. Why?

4. Also $\angle AOM = \angle BOM$, Why?

and $\therefore \angle AOD = \angle BOD$. Why?

5. $\therefore$ arc AC = arc CB, and arc AD = arc DB. § 85

253. COROLLARY 1. *A diameter which bisects a chord is perpendicular to it.*

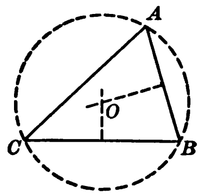
254. COROLLARY 2. *The perpendicular bisector of a chord passes through the center of the circle.*

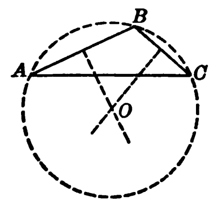
Discussion. Show how to bisect an arc whose center is not given.

255. A circle is said to be **circumscribed about a polygon** when the sides of the polygon are chords of the circle. The polygon is said to be **inscribed in the circle.**

PROPOSITION II. PROBLEM

256. *To circumscribe a circle about a given triangle.*





Given the triangle *ABC*.

Required *to circumscribe a ⊙ about the △ABC.*

Construction. 1. Construct the perpendicular bisectors of two of the sides and produce them until they meet in O (§ 241).

2. About O, with a radius equal to OB, describe a ⊙.

The ⊙ will pass through A, B, and C, since O is equidistant from $A, B,$ and C. (Why?) It is therefore circumscribed about the $\triangle ABC$, by the definition of a circumscribed circle. § 255

(Under what condition does the circumcenter lie on a side of the triangle? without the triangle?)

257. COROLLARY 1. *Only one circle may be circumscribed about a given triangle.*

For, if there were two such circles, they would intersect in three points, which is impossible (§ 71).

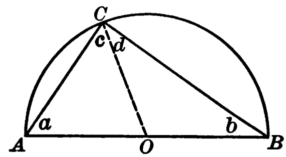
258. COROLLARY 2. *One and only one circle may be drawn through any three points not in the same straight line.*

259. COROLLARY 3. *A circle cannot be passed through three points in the same straight line.* § 68

260. COROLLARY 4. *The center of a given arc may be found by drawing two chords and finding the point of intersection of their perpendicular bisectors.*

261. A triangle is said to be **inscribed in a semicircle** when one of its sides is the diameter of the semicircle and the opposite vertex lies on the semicircle.

262. Corollary 5. *A triangle inscribed in a semicircle is a right triangle, the diameter being the hypotenuse.*



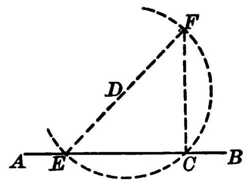
Proof. $OA = OB = OC$. Why?

$\therefore \angle a = \angle c$, and $\angle b = \angle d$. Why?

$\therefore \angle c + \angle d = 90°$. Why?

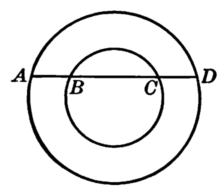
That is, the triangle is a right triangle, and the diameter is its hypotenuse.

263. Corollary 6. Problem. *At a given point in a given line to erect a perpendicular to that line.* (Second construction, see § 158.)



Given the line AB and the point C in it. Required to erect a $\perp$ to AB at C. Take any convenient point D not on AB, and with a radius DC describe a $\odot$ cutting AB in E. Draw the diameter through E, and let F be its other extremity. Draw CF. Then $CF \perp CE$. For $\triangle FCE$ is a right $\triangle$ with EF as its hypotenuse. (Why?)

EXERCISES



1. The perpendicular bisectors of the sides of an inscribed polygon are concurrent.

2. If, in the diagram, the two circles are concentric, prove that $AB = CD$.

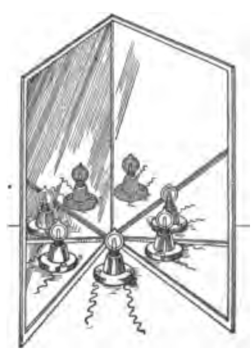
3. By means of a circular object (e.g. a coin) draw an arc of a circle. Locate its center.

4. A circle may be circumscribed about an isosceles trapezoid.

5. Two chords perpendicular to a third chord at its extremities are equal.

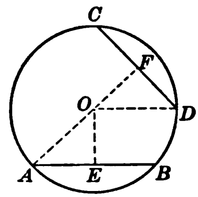
6. The radius of a circle circumscribed about an equilateral triangle is equal to two thirds the altitude.

7. If two plane mirrors are inclined at an angle and an object is placed within the angle, several images of the object will appear. These will lie on a circle. Explain.



PROPOSITION III. THEOREM

264. *In the same circle, or in equal circles, equal chords are equally distant from the center; and, conversely, chords equally distant from the center are equal.*



Given the circle O in which the chords AB and CD are equal.

To prove *that AB and CD are equally distant from the center.*

Proof. 1. Construct $OE \perp$ to AB, $OF \perp$ to CD, and draw the radii OA and OD.

(To be completed.)

Conversely, given the circle O, with AB and CD equally distant from the center O.

To prove *that* $AB = CD$.

Proof. (To be completed.)

EXERCISES

1. If the sides of an inscribed polygon are equally distant from the center, the polygon is equilateral.

2. Can a circle be circumscribed about any given parallelogram? Can a parallelogram be inscribed in a circle? Explain your answer.

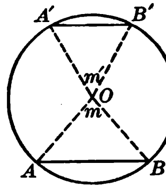
3. If two intersecting chords make equal angles with the diameter passing through the point of intersection, the two chords are equal.

4. A circle whose center is on the bisector of an angle cuts equal chords, if any, from the sides of the angle.

5. Two radii, OB and OC, of a circle intercept a minor arc. A point D in the arc is joined to the mid-points E and F of the radii. If $DE = DF$, prove that arc $CD =$ arc DB.

PROPOSITION IV. THEOREM

265. *In the same circle, or in equal circles, the greater of two minor arcs is subtended by the greater chord; and conversely, the greater chord subtends the greater minor arc.*



Given, in the circle *O*, the arc *AB* greater than the arc *A'B'*.

***To prove* that** *chord* $AB >$ *chord* $A'B'$.

Proof. 1. Draw the radii OA, OB, OA', OB'.

2. In the $\triangle s\ AOB$ and $A'OB'$,

$AO = A'O$, and $BO = B'O$. Why?

Also $\angle m > \angle m'$.

(Since arc $AB >$ arc $A'B'$, by Hyp.)

3. $\therefore AB > A'B'$. § 231

Conversely, given, in the circle *O*, the chord *AB* greater than the chord *A'B'*.

***To prove* that** *arc* $AB >$ *arc* $A'B'$.

Proof. 1. Draw the radii OA, OB, OA', OB'.

2. In the $\triangle s\ AOB$ and $A'OB'$,

$AO = A'O$, and $BO = B'O$. Why?

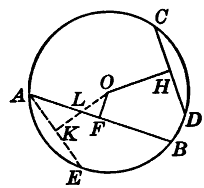
Also $AB > A'B'$. Hyp.

3. $\therefore \angle AOB > \angle A'OB'$. § 232

$\therefore$ arc $AB >$ arc $A'B'$. Why?

Proposition V. Theorem

266. *In the same circle, or in equal circles, if two chords are unequal, they are unequally distant from the center, and the greater chord is at the less distance.*



Given the chords *AB* and *CD* in the circle *O*, *AB* being greater than *CD*; given also *OF* perpendicular to *AB*, and *OH* perpendicular to *CD*.

***To prove** that* $OF < OH$.

Proof. 1. Minor arc $CD <$ minor arc AB. § 265

2. Take the point E on the minor arc AB, so that

arc $AE =$ arc CD.

Draw AE.

Draw $OK \perp$ to AE.

Then chord $AE =$ chord CD, Why?

and hence $OK = OH$. Why?

3. Now all points of the chord AE lie upon the opposite side of the chord AB from the center O.

$\therefore$ OK must intersect AB in some point, as L.

4. $\therefore OL < OK$. Why?

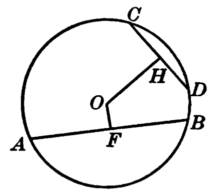
5. But $OF < OL$. Why?

$\therefore OF < OK$, Why?

or $OF < OH$.

Proposition VI. Theorem

267. *In the same circle, or in equal circles, if two chords are unequally distant from the center, they are unequal, and the chord at the less distance is the greater.*



Given the chords *AB* and *CD* in a circle *O*; also *OF*, the perpendicular to *AB*, less than *OH*, the perpendicular to *CD*.

***To prove** that* $AB > CD$.

Proof. 1. Now $AB = CD$, or $< CD$, or $> CD$.

2. If $AB = CD$, then $OF = OH$. Why?

But this is contrary to the hypothesis.

3. If $AB < CD$, then $OF > OH$. § 266

But this also is contrary to the hypothesis.

4. $\therefore AB > CD$.

EXERCISES

1. The shortest chord that can be drawn through a point within a circle is perpendicular to the diameter at that point.

2. The diameter is the greatest chord of a circle.

3. If a chord is drawn parallel to a diameter, the arcs intercepted between the chord and the diameter are equal. (Draw radii to the extremities of the chord.)

4. Two parallel chords intercept equal arcs on a circle (see Ex. 3).

5. If an arc is doubled, is the intercepting central angle doubled? Is the subtending chord also doubled? Why is it that a central angle is measured by its intercepted arc, while a chord is *not* measured by its subtended arc?

PROPOSITION VII. THEOREM

268. *The shortest line and also the longest line which can be drawn from a point to a circle lie on a line passing through the center of the circle and the given point.*

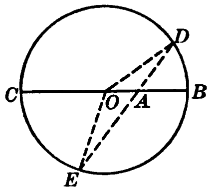


FIG. 1

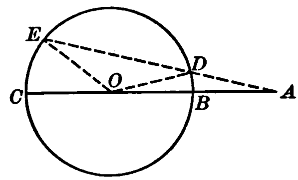


FIG. 2

Given the circle *O* and the point *A*, and *AB* and *AC* the shorter and longer segments respectively of a line drawn through *A* and passing through *O*, the center of the circle.

To prove *that AB is the shortest line, and AC the longest line, from A to the circle.*

CASE I. *When A is within the circle* (Fig. 1).

Proof. 1. Draw DE, any other line through A, cutting the circle in D and E. Let AE be its longer and AD its shorter segment.

Draw the radii OE and OD.

2. In the $\triangle OAD$, $OA + AD > OD$; Why?

that is, $OA + AD > OB$.

(Since $OB = OD$.)

Or $OA + AD > OA + AB$; that is, $AD > AB$. Ax. 10

Hence AB is less than any other line from A to the circle.

3. Also in the $\triangle OAE$, $OA + OE > AE$; Why?

that is, $OA + OC > AE$, or $AC > AE$. Why?

Hence AC is greater than any other line from A to the circle.

4. $\therefore$ AB is the shortest line, and AC the longest line, from A to the circle.

CASE II. *When A is without the circle* (Fig. 2).

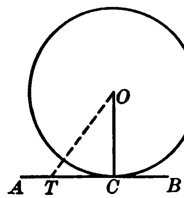
(To be completed.)

269. The **distance from a point to a circle** is the shortest line from the point to the circle.

TANGENTS

Proposition VIII. Theorem

270. *A straight line perpendicular to a radius at its extremity is a tangent to the circle.*



Given the circle *O*, the radius *OC*, and the line *AB* perpendicular to *OC* at its extremity *C*.

***To prove** that AB is tangent to the circle O.*

Proof. 1. From the center O draw any other line to AB, as OT.

2. Then $OT > OC$. Why?

3. $\therefore$ the point T is without the circle. Why?

4. Hence every point in the line AB, except C, lies without the circle, and therefore AB is tangent to the circle at C.

(Definition of a tangent, § 66.)

271. Corollary 1. *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

For OC is the shortest line from O to AB, and is therefore $\perp$ to AB.

272. Corollary 2. *A perpendicular to a tangent at the point of contact passes through the center of the circle.*

For the radius drawn to the point of contact is $\perp$ to the tangent, and therefore a $\perp$ erected at the point of contact coincides with this radius and passes through the center.

273. Corollary 3. *A perpendicular from the center of a circle to a tangent passes through the point of contact.*

PROPOSITION IX. PROBLEM

274. *Through a given point, to draw a tangent to a given circle.*

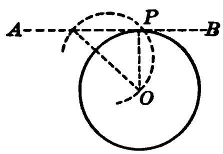


FIG. 1

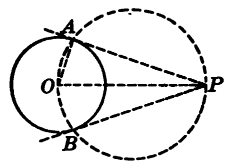


FIG. 2

Given the circle O and the point P.

Required *to construct a tangent to the circle O through the point P.*

CASE I. *When the given point P is on the circle O* (Fig. 1).

Construction. 1. Draw the radius OP.

2. Construct a line $AB \perp$ to OP at P. § 263

Then AB is the tangent required. Why?

CASE II. *When the given point P is without the circle* (Fig. 2).

Construction. 1. Draw the line OP, joining the given point and the center of the ⊙.

2. With OP as a diameter describe a circle intersecting the given circle at A and B.

3. Draw PA and PB. Each of these lines is tangent to the circle O.

Proof. 1. Draw OA. Then the $\triangle OAP$, being inscribed in a semicircle, is a right triangle. § 262

2. $\therefore PA \perp OA$. Why?

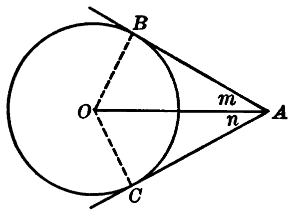
3. $\therefore PA$ is a tangent to the circle O. Why?

In the same manner, by drawing the radius OB, it may be proved that the line PB is a tangent to the circle O.

Discussion. **There are, therefore, *two* solutions of the above problem, when the given point is an *external* point.**

Proposition X. Theorem

275. *The tangents drawn to a circle from an external point are equal, and make equal angles with the line joining that point to the center.*



Given AB and AC tangent to the circle O; also the line OA.

***To prove** that* $AB = AC$, *and* $\angle m = \angle n$.

(To be completed.)

276. The chord connecting the points of contact of the two tangents to a circle from an external point is called the **chord of contact** of the tangents.

277. A circle is said to be **inscribed in a given polygon** when the sides of the polygon are tangent to the circle. The polygon is said to be **circumscribed about the circle.**

EXERCISES

1. In the above figure

What would be the relation of OA to the chord of contact BC?

Could the chord BC under any circumstances be the perpendicular bisector of OA?

What relation exists between $\angle BOC$ and $\angle BAC$?

If $\triangle BAC$ is equilateral, how many degrees in the $\angle BOC$?

If $\angle BAC$ is 120°, prove that $AB + AC = OA$.

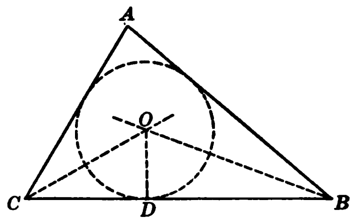
Prove that the quadrilateral $ABOC$ can be inscribed in a circle having OA for its diameter.

If $OA = 2\ OB$, how many degrees in the $\angle BAC$?

2. Construct a tangent to a given circle parallel to a given line; perpendicular to a given line. How many solutions are there?

PROPOSITION XI. PROBLEM

278. *To inscribe a circle in a given triangle.*



Given the triangle *ABC*.

Required *to inscribe a circle in* $\triangle ABC$.

Construction. 1. Bisect the angles B and C.

2. From O, the intersection of the bisectors, draw $OD \perp$ to BC.

3. About O as a center, with a radius equal to OD, draw a $\odot$. This is the $\odot$ required.

Proof. 1. O is equidistant from AB, AC, and BC. § 245

2. $\therefore AB$, AC, and BC are each $\perp$ to a radius of the above $\odot$ at its extremity. Why?

3. $\therefore AB$, AC, and BC are tangents to the $\odot$; that is, the $\odot$ is inscribed in the $\triangle ABC$. § 277

EXERCISES

1. If two circles are concentric, all chords of the outer circle which are tangents to the inner are equal to each other and are bisected at their points of contact.

2. A circle can be inscribed in a rhombus.

3. The sum of two opposite sides of a quadrilateral circumscribed about a circle is equal to the sum of the other two sides.

4. Conversely, if the sum of two opposite sides of a quadrilateral is equal to the sum of the other two sides, then a circle may be inscribed in the quadrilateral. (Prove by constructing a circle tangent to *three* sides of the quadrilateral, and showing that if the fourth side were to cut the circle or to lie wholly without the circle, it would lead to an absurdity.)

5. Given a circumscribed hexagon; prove that the sum of one set of alternate sides (first, third, fifth) equals the sum of the other set (second, fourth, sixth).

6. Is the statement of Ex. 5 true for other circumscribed polygons?

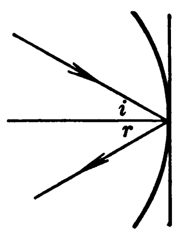
7. The radius of a circle inscribed in an equilateral triangle is equal to one third of the altitude.

8. The hypotenuse of a right triangle is equal to the sum of the legs diminished by twice the radius of the inscribed circle.

9. A parallelogram circumscribed about a circle is either a rhombus or a square.

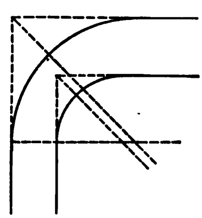
10. If from the extremities of a diameter of a circle perpendiculars are dropped upon a tangent, the sum of the perpendiculars is equal to the diameter.

11. When a ray of light strikes a spherical mirror (represented in cross section by a circle), the angle of incidence (see Ex. 32, p. 107) is found by drawing a tangent to the circle at the point of incidence, and erecting a perpendicular to the tangent at that point. In this case the perpendicular (called the **normal**) is a radius. (Why?)



The line of propagation of a *sound wave* also follows the law of reflection of a ray of light, namely, that the angle of incidence is equal to the angle of reflection.

The circular gallery in the dome of St. Paul's in London is known as a whispering gallery, for the reason that a faint sound produced at a point near the wall can be heard around the gallery near the wall, but not elsewhere. The sound is reflected along the circular wall in a series of equal chords. Explain why these chords are equal.



12. Two straight roads of different width meet at right angles. It is desired to join them by a road the sides of which are arcs of circles tangent to the sides of the straight roads. What construction lines are necessary? Draw such a figure.

13. A circle tangent to one side of a triangle and to the other two sides produced is called an **escribed** circle. How many escribed circles has a triangle? Draw them.

PROPOSITION XII. THEOREM

279. *Parallel lines intercept equal arcs on a circle.*

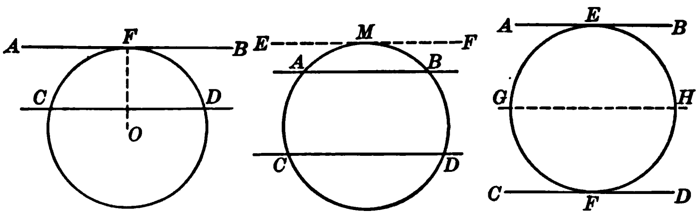


FIG. 1 FIG. 2 FIG. 3

CASE I. *When the parallels are a tangent and a secant.*

Given the circle O, with AB (Fig. 1), a tangent at F, parallel to CD, a secant.

To prove that arc $CF =$ arc DF.

Proof. 1. Draw the radius OF.

2. Then $OF \perp AB$. Why?

3. And hence OF is also $\perp$ to CD. Why?

4. $\therefore$ arc $CF =$ arc DF. Why?

(Case II, when both parallels are secants, and Case III, when both are tangents, are left as exercises.)

EXERCISES

1. A trapezoid inscribed in a circle is isosceles.

2. A circle has two parallel chords, which lie on opposite sides of the center. One is equal to the side of a regular hexagon, the other to the side of a regular decagon inscribed in the given circle. Find the number of degrees in the arcs intercepted between the chords.

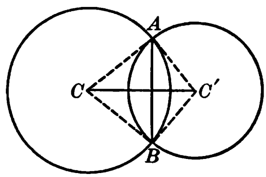
3. Solve Ex. 2, if the two chords are respectively sides of a pentagon and an octagon; a pentagon and a dodecagon; a hexagon and an octagon; a square and a decagon; a triangle and a hexagon.

4. Solve Exs. 2 and 3, if the parallel chords are on the same side of the center.

TWO CIRCLES

Proposition XIII. Theorem

280. *If two circles intersect each other, the line of centers is perpendicular to their common chord at its middle point.*



Given two circles whose centers are C and C', intersecting at the points A and B, the common chord AB, and the line of centers CC'.

To prove *that* CC' is $\perp$ to AB *at its middle point.*

Proof. 1. Draw CA, CB, $C'A$, and $C'B$.

2. Then $CA = CB$, and $C'A = C'B$. Why?

3. $\therefore$ CC' is the perpendicular bisector of AB. Why?

281. Two circles are tangent to each other if both are tangent to a straight line at the same point. The circles are said to be tangent *internally* or *externally*, according as one circle lies wholly *within* or *without* the other. The common point is called the **point of contact.**

EXERCISES

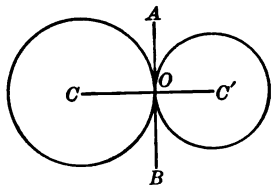
1. Describe the relative position of two circles, if the line-segment joining their centers is equal to the sum of the radii; is equal to the difference of the radii.

2. Given the line l and the point P on that line. Construct two circles tangent to l at P, each with a radius equal to another line m.

3. Given two intersecting lines and a point P on one of those lines. Construct two circles tangent to both lines and passing through P.

Proposition XIV. Theorem

282. *If two circles are tangent to each other, the line of centers passes through the point of contact.*



Given the circles C and C' tangent to the straight line AB at O, and the line of centers CC'.

To prove *that the line CC' passes through O.*

Proof. 1. A line ⊥ to AB at the point O will pass through the centers C and C'. Why?

2. Hence the line CC' must coincide with that ⊥. Why?

3. ∴ O is in the straight line CC'.

EXERCISES

1. PROBLEM. To construct three circles about the vertices of a given triangle as centers, each circle being tangent externally to the other two.

Suggestion. Inscribe a circle in the given triangle.

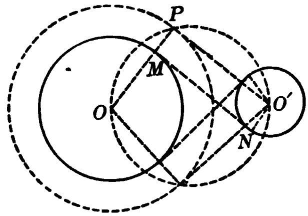
2. Find the radii of three such circles, if the sides of the triangle are 4, 5, and 6 (8, 8, and 14).

3. In accurate tool work, where holes are to be bored near each other in a metal plate, the centers are first marked carefully to thousandths of an inch. This is often done by first turning disks on a lathe, the diameters of the disks being such that when they are placed tangent to each other their centers will mark the positions of the centers of the holes to be bored.

Three holes are to be bored. The distances between their centers are 0.650 in., 0.790 in., and 0.865 in. respectively. Find the diameters of the required disks. (From "School Science and Mathematics.")

PROPOSITION XV. PROBLEM

283. *To draw a common tangent to two given circles.*



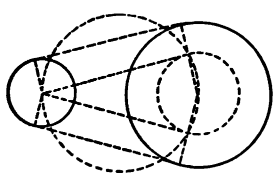
CASE I. *When the tangent is to cross the line-segment joining the centers (common internal tangent).*

Given the circles *O* and *O′*.

***Required** to construct a common internal tangent to the circles O and O′.*

Analysis. It is apparent that such a tangent, when drawn, would be ⊥ to the radii *OM* and *O′N* drawn to *M* and *N*, the points of tangency. Now if through *O′* a line be drawn ∥ to the common tangent, cutting *OM* produced in *P*, *MNO′P* would be a rectangle, *MP* would be equal to *NO′*, and *OP* would be equal to *OM* + *O′N*, or the sum of the radii; also, *O′P* would be tangent to a ⊙ about *O* with a radius equal to *OP*, that is, equal to *OM* + *O′N*.

(Construction and proof to be completed.)



CASE II. *When the tangent is not to cross the line-segment joining the centers (common external tangent).*

Analysis. The conditions of the problem in this case are identical with those of Case I, except that the radius *OP* will be equal to the difference of the radii *OM* and *O′N*

(Construction and proof to be completed.)

EXERCISES

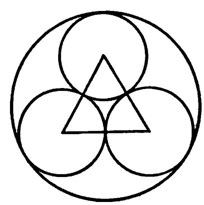
1. If two common external tangents or two common internal tangents are drawn to two circles, the segments intercepted between the points of contact are equal.

2. How should two wheels be connected (a) by belting, and (b) by gearing, so as to revolve in the same direction? in opposite directions?

3. In each of the different positions of two circles mentioned in Ex. 1, p. 39, how many common internal and how many common external tangents may be drawn?

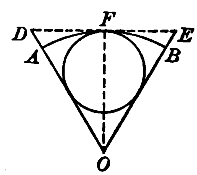
4. Construct the common tangents to two equal circles which do not intersect.

5. About the vertices of an equilateral triangle as centers, with radii equal to one half the side of the triangle, draw circles. These circles are tangent to each other. (Why?) Show how to construct a circle which shall inclose these circles and be tangent to all three of them. (This construction is the basis of a design which appears frequently in decoration.)

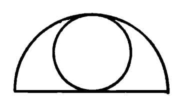


6. Construct a circle which shall be tangent to a given arc, and to the radii drawn to the ends of the arc.

(If a tangent is drawn at the mid-point of the arc, show that the required circle will be inscribed in the triangle formed by this tangent and the radii produced.)



7. Arcs of circles, either alone or combined with their chords, are used to form arches, which appear frequently in decorative design. A common form is the semicircular arch, which consists of a semicircle and its diameter. Inscribe a circle in such an arch as shown in the figure. Would the same method of construction hold good if the arc were less than a semicircle?



8. Apply Ex. 6, (a) to inscribe in a given semicircular arch two equal tangent circles; (b) to draw in a semicircular arch three equal circles tangent to the semicircle, of which two shall be tangent to the diameter, and the third tangent to the other two circles.

9. The spiral shown in Fig. 1 is composed of semicircles, having as centers the points A and B alternately. Draw such a spiral, and prove that two consecutive semicircles are tangent to each other.

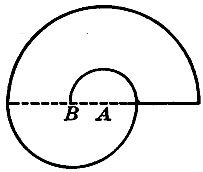


FIG. 1

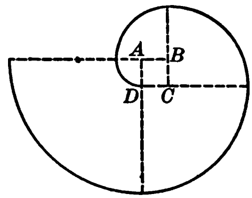
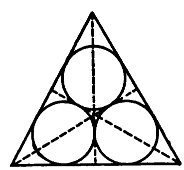


FIG. 2

10. A four-centered spiral may be constructed by extending in succession the sides of a square and using each vertex as the center of a quadrant, as shown in the figure. Draw such a spiral (Fig. 2).

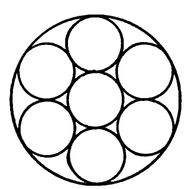
11. Construct in similar fashion a three-centered spiral of three arcs; a six-centered spiral of six arcs.



12. Three circles are drawn in an equilateral triangle, such that each is tangent to the other two circles and to two sides of the triangle. Show that each circle is tangent to two altitudes of the triangle, and give a method for making the construction.

13. Within a given square construct four equal circles such that each is tangent to two others and to two sides of the square.

(Divide the square into four smaller squares and inscribe a circle in each. Show that these circles answer the required conditions.)



14. Within a given square construct four equal circles such that each is tangent to two others and to one side of the square.

(Draw the diagonals of the square.)

15. Construct the design shown in the figure.

(Note that the radius of each of the small circles is one third the radius of the large circle. Three of the small circles lie along one diameter. How are the centers of the other four small circles found?)

ANGLE MEASUREMENTS

284. An angle is inscribed in a circle when its vertex is on the circle and its sides are chords.

An angle is inscribed in an arc when its vertex lies on the arc and its sides pass through the ends of the arc.

PROPOSITION XVI. THEOREM

285. *An inscribed angle is measured by one half the arc intercepted between its sides.*

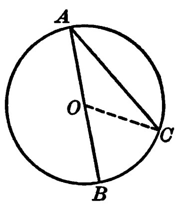


FIG. 1

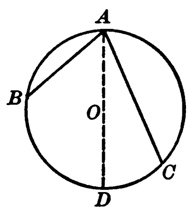


FIG. 2

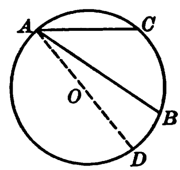


FIG. 3

Given the circle *O*, with the angle *BAC* inscribed in it.

To prove *that* $\angle BAC$ *is measured by* $\frac{1}{2}$ *arc BC.*

CASE I. *When the center O is in one of the sides of the angle* (Fig. 1).

Proof. 1. Draw OC.

2. Then $\angle A = \angle C$. Why?

3. But $\angle BOC = \angle A + \angle C$. Why?

$\therefore \angle BOC = 2\angle A.$

4. But $\angle BOC$ is measured by arc BC. § 94

(A central angle is measured by its intercepted arc.)

5. $\therefore \angle A$ is measured by $\frac{1}{2}$ arc BC. Ax. 5

(Cases II and III, when the center falls within or without the angle, to be completed.)

REMARK. This theorem is otherwise stated: "The number of degrees in an inscribed angle is equal to one half the number of arc-degrees in the intercepted arc" (see § 92).

286. COROLLARY 1. *An angle inscribed in a semicircle is a right angle.*

For it is measured by half a semicircle (Fig. 4). See also § 262.

287. COROLLARY 2. *An angle inscribed in an arc greater than a semicircle is an acute angle.*

For it is measured by one half an arc less than a semicircle; as $\angle A$ (Fig. 5).

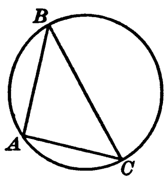


FIG. 4

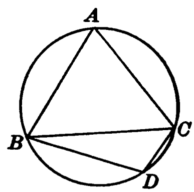


FIG. 5

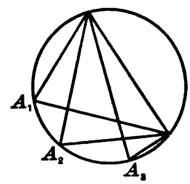


FIG. 6

288. COROLLARY 3. *An angle inscribed in an arc less than a semicircle is an obtuse angle.*

289. COROLLARY 4. *Angles inscribed in the same arc, or in equal arcs of the same circle, are equal* (Fig. 6).

EXERCISES

1. Test by Proposition XVI the following theorems:

(a) The sum of the angles of a triangle equals two right angles.

(b) If two sides of a triangle are equal, the angles opposite are equal.

(c) If two angles of a triangle are equal, the sides opposite are equal.

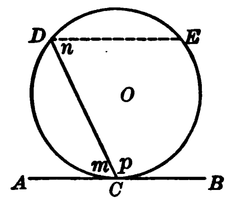
These tests cannot be regarded as proofs. Why?

2. If two triangles have their angles respectively equal and are inscribed in the same circle, they are congruent.

3. The bisectors of all angles inscribed in the same arc are concurrent.

Proposition XVII. Theorem

290. *An angle included by a tangent and a chord drawn from the point of contact is measured by one half the intercepted arc.*



Given the angle m included by AB, a tangent to the circle O at C, and the chord CD.

To prove that $\angle m$ *is measured by* $\frac{1}{2}$ *arc* CD.

Proof. 1. Draw $DE \parallel$ to AB through D.

2. Then arc $CD =$ arc CE. Why?

3. Also $\angle m = \angle n$. Why?

4. But $\angle n$ is measured by $\frac{1}{2}$ arc CE. Why?

$\therefore \angle m$ is measured by $\frac{1}{2}$ arc CD. Ax. 1.

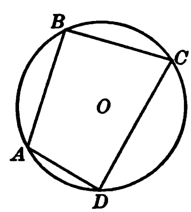
5. Also $\angle p$ is measured by $\frac{1}{2}$ major arc CED.

(Being the supp. of $\angle m$, while arc CED is the conjugate of arc CD.)

EXERCISES

1. The opposite angles of an inscribed quadrilateral are supplementary.

(By which arcs are $\angle s$ A and C measured? What is the sum of these arcs?)

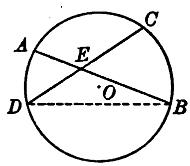


2. If the opposite angles of a quadrilateral are supplementary, the quadrilateral may be inscribed in a circle (converse of Ex. 1).

Suggestion. Pass a circle through three of the vertices (§ 258), and show by the Indirect Method that the fourth vertex must lie on the circle.

PROPOSITION XVIII. THEOREM

291. *An angle formed by two chords intersecting within the circle is measured by half the sum of the intercepted arcs.*



Given the circle *O*, in which two chords, *AB* and *CD*, intersect in *E*.

To prove *that* $\angle AED$ *is measured by* $\frac{1}{2}$ *(arc* AD + *arc* CB*).*

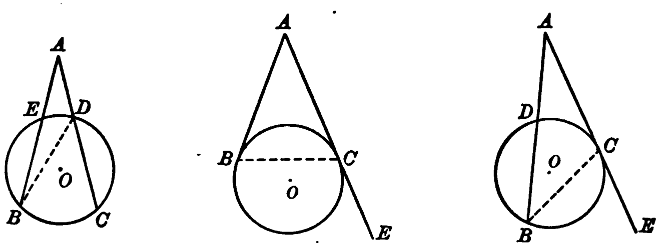
Proof. 1. Draw DB.

2. Then $\angle AED = \angle B + \angle D$. Why?

(To be completed.)

PROPOSITION XIX. THEOREM

292. *An angle formed by two secants, or two tangents, or a tangent and a secant, intersecting without a circle, is measured by half the difference of the intercepted arcs.*



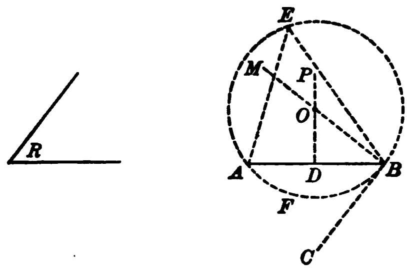
Given the circle *O* with two secants, or two tangents, or a tangent and a secant intersecting at *A* without the circle.

To prove *that* $\angle A$ *is measured by half the difference of the intercepted arcs.*

(Cases I, II, and III to be completed.)

PROPOSITION XX. PROBLEM

293. *Upon a given straight line as a chord to construct an arc of a circle in which a given angle may be inscribed.*



Given the straight line *AB* and the angle *R*.

Required *to construct, on the line AB as a chord, an arc of a circle such that the angle R may be inscribed in the arc.*

Construction. 1. Construct the $\angle ABC$ equal to $\angle R$.

2. Draw DP the $\perp$ bisector of AB.

3. At B construct $BM \perp$ to BC.

4. About O, the point of intersection of DP and BM, as a center, with a radius equal to OB, draw a circle. This circle will pass through A, and the arc AEB is the arc required.

Proof. 1. The point O is equidistant from A and B. Why?

$\therefore$ the circle passes through A. Why?

2. But $BC \perp BO$. Cons.

$\therefore BC$ is tangent to the $\odot$. Why?

3. Then $\angle ABC$ is measured by $\frac{1}{2}$ arc AFB. Why?

4. But any angle, as $\angle AEB$, inscribed in the arc AEB, is also measured by $\frac{1}{2}$ arc AFB. Why?

5. $\therefore$ any angle inscribed in the arc AEB equals $\angle R$.

NUMERICAL EXERCISES

1. An inscribed angle intercepts an arc of 40° (50°, 60°, 75°). How many degrees in the angle?

2. An angle of 20° (30°, 45°, 50°, 67° 30′) is inscribed in a circle. How many degrees in the intercepted arc?

3. The sides of an inscribed triangle subtend arcs of 100°, 120°, and 140°. How many degrees in each angle of the triangle?

4. The arcs subtended by the sides of an inscribed triangle are in the ratio 1 : 2 : 3. What kind of triangle is it?

5. The sides of an inscribed quadrilateral subtend arcs in the ratio 1 : 2 : 3 : 4 (3 : 5 : 7 : 9). How many degrees in each angle of the quadrilateral?

6. In the preceding exercise how many degrees in the angles between the diagonals of the quadrilateral?

7. A triangle is inscribed in a circle, and another triangle is circumscribed by drawing tangents at the vertices of the inscribed triangle. The angles of the inscribed triangle are 40°, 60°, and 80°. Find all the other angles of the figure.

8. The vertex angle of an inscribed isosceles triangle is 100°. How many degrees in the arcs subtended by each of the sides?

9. The bases of an inscribed isosceles trapezoid subtend arcs of 100° and 120°. How many degrees in each angle of the trapezoid (a) if the bases are on the same side of the center? (b) if they are on opposite sides of the center?

10. At the vertices of an inscribed quadrilateral tangents are drawn to the circle, forming a circumscribed quadrilateral. The arcs subtended by the sides of the inscribed quadrilateral are in the ratio 3 : 4 : 5 : 8. Find

(a) the angles of each quadrilateral;

(b) the angles between the diagonals of the inscribed quadrilateral;

(c) the angles between the opposite sides of the inscribed quadrilateral produced to intersect;

(d) the angles between the sides of the inscribed and those of the circumscribed quadrilateral.

11. A regular hexagon is inscribed in a circle and all its diagonals are drawn. How many degrees in each angle of the figure?

12. The angle formed by two tangents drawn to the same circle is 100°. How many degrees in the two arcs subtended by the chord of contact?

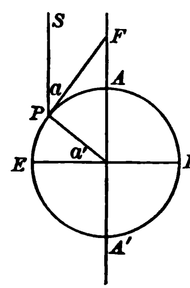
13. If the angle between two tangents to the same circle is 60°, what kind of triangle is formed when the chord of contact is drawn?

14. Given two tangents to a circle; the major arc contains 200°. How many degrees in the angle formed by the tangents?

15. The sides of an inscribed pentagon subtend arcs each of which is 10° greater than the preceding one. Draw the diagonals of the pentagon and determine each angle of the figure.

16. In the preceding exercise draw tangents at the vertices of the inscribed pentagon and determine each of the interior angles of the resulting circumscribed pentagon.

17. The diagram shows how the latitude of a place may be determined by observation of the pole star. Let EE' represent the equator, AA' the axis of the earth, P the place whose latitude is to be found, PF a plane tangent to the earth at P (the horizon), and PS the line of observation of the pole star. Then $\angle a'$ represents the latitude, and $\angle a$ is called the altitude of the pole star. For practical purposes we may assume that $AA' \parallel PS$.



Prove that $\angle a' = \angle a$.

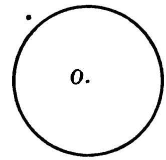
18. An inscribed angle is formed by the side of a regular inscribed hexagon and the side of a regular inscribed decagon. How many degrees in the angle? (Give two solutions.)

19. Solve Ex. 18 if the inscribed angle is formed by sides of the following regular polygons:

(a) triangle and square;
(b) pentagon and octagon;
(c) hexagon and octagon;
(d) pentagon and dodecagon.

LOCI

294. The definition of a circle may be used to introduce a very important geometric idea. All points in a plane which are two inches from a given point O lie in a circle whose center is the given point O and whose radius is two inches long.



Conversely, every point in this circle is two inches from O.

These two statements are sometimes replaced by the one statement that the circle about O is the **locus** (that is, *place*) of the points two inches from O. In general,

The **locus of a point satisfying a given condition** is the figure containing all the points that fulfill the given condition (or answer the given description), and no other points.

Hence the following

295. Rule for solving Locus Problems:

1. Locate a number of points which satisfy the given condition, and thus obtain a notion of what the locus is.

2. Prove that every point satisfying the given condition lies in the assumed locus.

3. Prove that every point of the assumed locus satisfies the given condition.

In the following simple exercises, however, the answers may be stated without proof.

EXERCISES

1. Where are all the houses that are 1 mi. from your school building?

2. Where are all the villages that are 10 mi. from your own town?

3. Sound travels about 1100 ft. per second. If a cannon is fired from a certain point, what is the locus of all persons who hear the report after 3 sec.?

4. A ladder leans against a wall. A man stands on the middle rung. If the ladder slips down, what is the locus of the man's feet?

5. What locus problems are suggested to you by the opening of a book or a door? by a seesaw? a pendulum? the governor of an engine? a clock?

Theorem I. The locus of a point X at a given distance d from a given point P is a circle having P as its center and d as its radius.

EXERCISES

1. What is the locus of a house 100 ft. from a straight road?

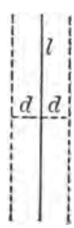
2. On a city map what is the locus of a house 200 ft. from the "mile circle"?

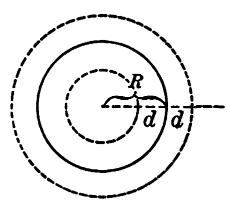
3. Where are all the points that are 2 in. from the surface of the table?

4. What is the locus of the foot of a tree which is 10 ft. from the wall of a round tower?

5. What is the locus of a point 1 in. from the surface of a spherical shell?

Theorem II. The locus of a point X at a given distance d from a given line l consists of two lines parallel to l, one on each side of l, and d units from it (§ 237).





Theorem III. The locus of a point X at a given distance d from a given circle whose radius is R consists of two circles concentric with the given circle and with radii $R + d$ and $R - d$ respectively (§ 269).

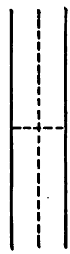
EXERCISES

1. A motor boat sails up a straight canal midway between the banks. What is the locus of the boat?

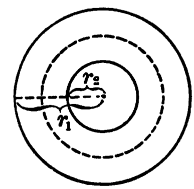
2. The hands of a clock describe concentric circles. What is the locus of a point equidistant from these circles?

3. What is the locus of a point equidistant from two opposite walls of a rectangular room?

Theorem IV. The locus of a point equidistant from two parallel lines is a parallel line midway between the two parallels (§ 237).



Theorem V. The locus of a point equidistant from two concentric circles is a concentric circle midway between those two circles (§ 269).



What is the radius of the locus if the radii of the given circles are r_1 and r_2?

EXERCISES

1. The poles of a telephone line are to be each equidistant from two houses. What is the locus of these poles?

2. What is the answer in Ex. 1, if the poles of the telephone line are to be equidistant from two intersecting straight roads?

Theorem VI. The locus of a point equidistant from two given points is the perpendicular bisector of the line joining the two points (§ 239).

Theorem VII. The locus of a point equidistant from two intersecting straight lines consists of the pair of straight lines that bisect the angles formed by the two given lines (§ 240).

EXERCISES

1. In front of a rectangular house, trees are planted in such a manner that the lines joining the foot of each tree to the two nearest corners of the house inclose a right angle. Where are the trees situated?

2. How would the trees be located, if the lines joining the foot of each tree to the two nearest corners of the house inclosed an angle of 40°?

3. Given two fixed points A and B. Through A a line is drawn perpendicular to a line passing through B. Call their point of intersection C. What is the locus of C?

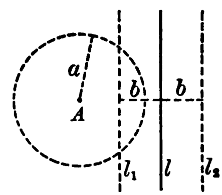
4. State the locus theorems underlying Exs. 1–3 (§ 293).

296. Intersection of loci. A point sometimes fulfills more than one condition. The solution then consists in finding the points common to two or more loci.

EXAMPLE

Find a point X, a inches from a given point A and b inches from a given line l.

The solution is readily obtained by applying the Theorems I and II on page 172. The points of intersection of the line l_1 and the circle in the figure evidently satisfy both conditions.



Discussion. What is the result (1) if l_1 touches the circle? (2) if the circle cuts l_1 and l_2? (3) if A is on l?

In what way do the relative values of a and b affect the result?

EXERCISES

1. Construct a point 2 in. from a given point and 3 in. from a given line. When is the solution impossible?

2. Find a point P which lies in a given line and is equidistant from two given points. Discuss the problem.

3. Construct a point equidistant from two intersecting lines and also at a given distance from one of the lines.

4. Construct a point equidistant from the sides of an angle and at a given distance from the vertex.

5. A tree is to be planted 10 ft. from the front wall and 15 ft. from a corner of a rectangular house. How many solutions are possible?

6. Find in a given circle a point which is equidistant from two given points.

7. Construct a point equidistant from two given concentric circles and also equidistant from two given lines.

8. Find the locus of the center of a circle which (a) passes through two given points; (b) touches each of two given parallels; (c) touches a given line AB at a given point P.

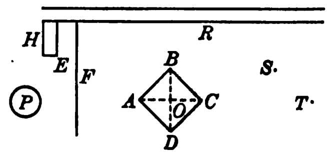
9. Construct a circle which has its center in a given line and passes through two given points.

10. Construct a circle which passes through two given points and has its center equidistant from two given intersecting lines.

11. Construct a point which is at a given distance from a given circle and at a given distance from a secant of that circle.

12. Construct a point which is equidistant from two given parallel lines and from a transversal to those lines.

13. Construct a circle which is tangent to a given line at a given point and has its center on a given line.



14. In the diagram let $ABCD$ represent a baseball diamond (square); R, a street; H, a house; E, a corner of the house; S and T, two trees; F, a fence; P, a circular pond. How would you locate a person who is (a) equidistant from S and T and 5 yd. from CD? (b) 10 yd. from R and 3 yd. from E? (c) 3 ft. from P and 30 yd. from O? (d) equidistant from R and F? (e) equidistant from AB and CD, and from S and T? (f) 20 yd. from AB and 5 ft. from F?

MISCELLANEOUS EXERCISES ON LOCI

1. What is the locus of the vertex of a triangle that has a given base and a given altitude?

2. What is the locus of the vertex of the right angle of a right triangle which has a given hypotenuse?

3. Find the locus of the vertex of a triangle which has a given base and a given vertex angle.

4. What is the locus of the center of a circle which is inscribed in a triangle with a given base and a given vertex angle?

5. What is the locus of the mid-point of a line which is drawn from a given point to a given line?

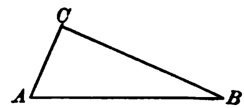
6. What is the locus of the mid-point of a chord of given length in a circle of given radius?

7. What is the locus of the mid-points of all chords drawn from a given point in a given circle?

8. What is the locus of the point of intersection of the diagonals of a rhombus constructed on a given line as a base?

9. What is the locus of a point equidistant from two given equal circles?

10. A square (equilateral triangle, regular hexagon) is moved along a straight line by revolving it successively about its vertices. What is the locus of one vertex of the square (triangle, hexagon) from one point of contact with the line to its next successive point of contact?



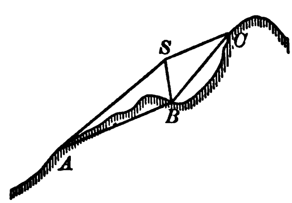
11. Two stations, A and B, on the shore of a lake are 900 yds. apart. A ship C is observed simultaneously from both stations, and the angles CAB and CBA are measured by the observers at each station. In order to determine the path of the ship, additional measurements are made from time to time and are telephoned from one station to the other. Determine the path of the ship if the measurements are as follows:

	$\angle CAB$	$\angle CBA$		$\angle CAB$	$\angle CBA$
(1)	50°	40°	(2)	71°	69°
	55°	35°		80°	60°
	59°	31°		90°	50°
	70°	20°		100°	40°

12. How may the accuracy of a drawing of a circle be tested with a carpenter's square?

13. What locus problem is suggested by a "saw-toothed" roof?

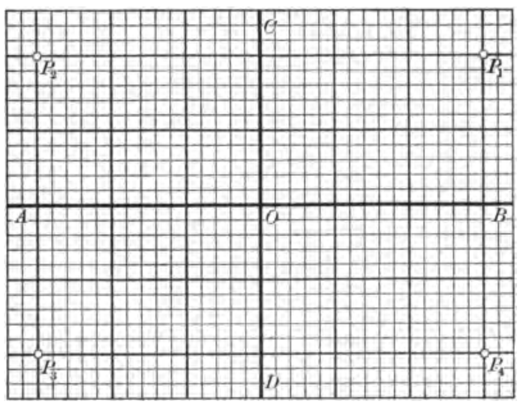
14. Three fortified islands, A, B, C, are so far from land that their guns do not carry to shore. A hostile cruiser is reconnoitering in the vicinity of the islands and the captain wishes to sail around them by the shortest possible route, but always out of range of the guns. Construct the course of the cruiser, using arbitrary values for the ranges of the guns on the islands.



15. A ship, S, approaching land is in the vicinity of three prominent landmarks, A, B, C. The captain wishes to determine how far he is from the shore. He finds that the distance AB subtends at S an angle of 60°, while BC subtends an angle of 95°. From the map the captain finds that $AB = 16$ mi., and $BC = 12$ mi. Explain how the length of SB can be found by construction and measurement. (The "three-point problem.")

COÖRDINATES. SQUARED PAPER

297. Draw $AB \perp$ to CD. From the intersection point O lay off equal segments on the four rays OA, OB, OC, OD. Through the points of division draw perpendiculars to the lines. A network of squares will be formed. Paper showing such an arrangement of squares is called **squared paper.** "Inch paper" is ruled into square inches and tenths of an inch; "millimeter paper" is ruled into square millimeters and square centimeters.



The point P_1 in the figure is represented numerically by the expression (15, 10); this means that to reach the point P_1 from O, the pencil's point should travel 15 units to the right along OB, and 10 units upward along a line parallel to OC. This process of locating a point by means of two numbers is called **plotting the point.** The point O is called the **origin.** The lines AB and CD are called **axes.** The two numbers are called the **coördinates** of the point. The distance from O on AB is called the **abscissa,** and the distance from the line AB is called the **ordinate** of the point. Distances measured to the right from O on OB are considered as positive (+) numbers, and distances to the left from O on OA are considered as negative (–) numbers. Distances vertically upward from AB are considered as positive, while distances vertically downward from AB are considered negative. Hence the coördinates of P_2 are (–15,10); those of P_3 are (–15, – 10); those of P_4 are (15, –10).

EXERCISES

1. Locate on a piece of squared paper four points, and name their coördinates with reference to two convenient axes.

2. Locate the points (5, 6), (– 4, 7), (– 8, – 6), (7, – 5).

3. Plot the following points and connect them so as to form a pattern: (0, 0), (0, 2), (0, 5), (0, 7); (2, 0), (2, 2), (2, 5), (2, 7); (5, 0), (5, 2), (5, 5), (5, 7); (7, 0), (7, 2), (7, 5), (7, 7).

4. Plot the points (4, – 5), (– 4, – 5), (0, 8). What kind of geometrical figure is determined by these points?

5. What figure is determined by the points (10, 7), (6, – 3), (– 6, – 3), (– 2, 7)?

6. A rectangle drawn on squared paper is symmetrical with respect to both axes. One vertex is (12, 5). What are the other vertices?

7. Determine, by drawing, the coördinates of the points which are 5 units from the horizontal axis, and at a distance from the origin equal to 13 units.

8. Ascertain, by drawing, the coördinates of the points which are equidistant from the two axes, and at a distance from the origin equal to 10 units.

298. The method of coördinates may be conveniently used to determine the locus of a point when the prescribed condition involves the distances of the point from two perpendicular lines.

EXAMPLES

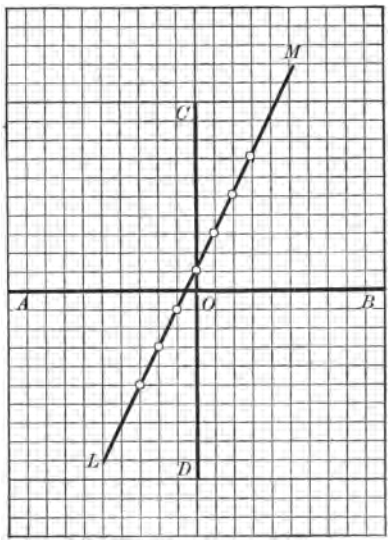
1. The distances x and y of a point from two given perpendicular lines satisfy the equation

(1) $$y = 2x + 1.$$

Determine the locus of the point.

Solution. We interpret x and y as coördinates with respect to the given lines as axes. To locate a number of points which satisfy the given condition (1), assume values for x and compute from (1) the corresponding values of y. For example, taking $x = 2$, then $y = 2 \times 2 + 1 = 5$. Hence the point (2, 5) is on the locus. In this way we compute the coördinates of any desired number of points lying on the locus. The results may be conveniently arranged in the following table:

When $x =$	-3	-2	-1	0	1	2	3	etc.
then $y =$	-5	-3	-1	1	3	5	7	etc.



Now plot the successive points $(-3, -5)$, $(-2, -3)$, $(-1, -1)$, etc., of this table. All of these points lie on the locus, since they satisfy the prescribed condition. They appear to lie on a straight line. Draw this line *LM*. To prove that this line is the locus, that is, to verify 2 and 3, § 295, would involve methods which the student would not now understand. It must suffice here to state the fact that the straight line *LM* is the required locus.

In algebra the straight line *LM* is called the "graph" of the equation (1).

Observe that (1) is an equation of the first degree in x and y. It may be shown that the locus is always a straight line if the coördinates satisfy an equation of the first degree.

2. The sum of the distances of a point from two perpendicular lines equals 6. What is the locus?

Solution. Taking the perpendicular lines as axes and the distances as coördinates, the given condition may be written

$$x + y = 6. \tag{2}$$

The locus is now determined as in Ex. 1, and is a straight line.

EXERCISES

1. The distances x and y of a point from two perpendicular lines satisfy one of the following equations. Determine the locus in each case:

(a) $y = 3x - 1$. (c) $3x + y = 5$. (e) $4x + 3y = 12$.
(b) $x = 2y - 6$. (d) $x - 2y = 5$. (f) $2x - 3y = 6$.

2. The difference of the distances of a point from two perpendicular lines equals 4. What is the locus?

3. What is the locus of a point whose abscissa is 6 units greater than the ordinate?

4. The distance of a point from one of two perpendicular lines equals twice its distance from the other. Draw the locus.

5. If the coördinates of a point satisfy an equation which is not of the first degree, the locus will usually be a curved line. This is the case in the following examples. Draw the locus in each case:

(a) $2y = x^2$. (b) $xy = 3$. (c) $y^2 = 4x$.

6. If p is the perimeter and s a side of any square, then $p = 4s$. Draw the graph of this equation, plotting values of s as abscissas and the corresponding values of p as ordinates.

7. The perimeter of an isosceles triangle is 12. If x is the base and y one of the legs, write the equation satisfied by x and y and draw a graph of it.

8. The graphic method may be used to illustrate geometric relations even when the equation showing the relation is not given. For example, draw a circle of radius 2 in. By means of the protractor mark off on this circle from some point A arcs of 10°, 20°, 30°, 40°, etc., to 180°. Draw from A the chords of these arcs. Using the number of degrees in the arcs as abscissas, and the measured lengths of corresponding chords as ordinates, plot the points thus determined. Is the graph a straight or a curved line? Continue the graph by increasing the number of the arcs at intervals of 10° up to 360°. What change is there in the corresponding chords?

9. Plot a graph showing the relation between the altitude of an equilateral triangle and a side. Use the bases of a series of equilateral triangles as abscissas, and the corresponding altitudes as ordinates.

CONSTRUCTION OF GEOMETRIC FIGURES

299. The theorems of Book II and the principles of loci afford additional methods of construction of geometric figures.

300. Determining Parts of a Figure. A man who wishes to build a house consults an architect with regard to plans. It is not necessary for the builder to give the architect all the dimensions of all parts of the house. Certain measurements furnished by the builder enable the architect to complete the plans for a building answering all requirements.

In like manner it appears that only a certain number of parts of a geometric figure are necessary to determine the figure. These are known as *determining parts.*

TRIANGLES

301. The six **principal parts** of a triangle are the three sides and the three angles.

How many of these parts are necessary to determine a triangle? Is the choice of parts otherwise limited?

302. In any triangle there are, in addition to the principal parts above named, other lines and angles which may serve to determine the figure, such as the medians, the altitudes, the bisectors of the interior angles, the angles between these lines, and the radii of the circumscribed and inscribed circles. These lines and angles are called **secondary parts.**

In the triangle ABC (p. 182) let a, b, and c represent the sides opposite the angles A, B, and C respectively.

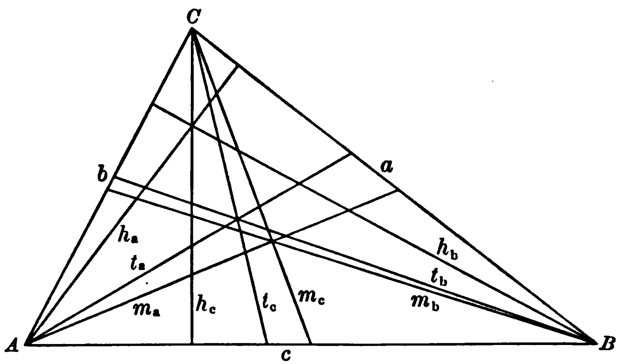
Let m_a, m_b, and m_c represent the medians on a, b, and c respectively.

Let h_a, h_b, and h_c represent the altitudes on a, b, and c respectively.

Let t_a, t_b, and t_c represent the bisectors of the angles opposite a, b, and c respectively.

Let R and r represent the radii of the circumscribed and inscribed circles respectively.

It will appear that, in general, three parts are necessary and sufficient to determine a triangle, provided one of these parts is a line. But the choice of parts and of values of these parts is limited to some extent by their relation to one another.



303. Relation of Parts. These relations make necessary a discussion of the limits within which a solution of a given problem is possible.

Try to construct a triangle whose sides are 5, 8, and 14 units respectively. Why is this construction impossible?

Try to construct a triangle two of whose angles are 80° and 110°. Why is this construction impossible?

How does the altitude on one side of a triangle compare in length with each of the other two sides? with the corresponding median? with the bisector of the corresponding interior angle?

304. Method of Solution. When an architect has had submitted to him certain requirements for a building, his practice generally is, first to form a mental image of the structure as it would appear if completed; then to make a rough sketch embodying his idea; then to indicate on this sketch the given parts; then to ascertain what other parts are determined by what is given him; and finally to make an accurate drawing with the aid of the information he has thus obtained.

305. A somewhat similar course of procedure is indicated by the rule on the opposite page.

RULE FOR THE SOLUTION OF CONSTRUCTION PROBLEMS

1. *Imagine the construction completed, and make a rough sketch of the figure.*

2. *Indicate on this sketch the given parts.*

3. *Ascertain what other parts are determined by means of the given parts.*

4. *If necessary, add such construction lines to the rough sketch as will make some part of the figure fully determined.*

5. *Begin the actual construction of the figure with that part of it, usually a triangle, which is fully determined.*

6. *Complete the figure in accordance with the general principles of construction lines* (§§ 145 *seq.*).

7. *Prove that the construction satisfies the given conditions, or else show that such proof is unnecessary.*

8. *Discuss the limits within which the solution is possible, and the number of possible solutions when there is more than one, as sometimes happens.*

306. Determination of Triangles. It follows from the laws of congruence of triangles that a triangle is uniquely determined (only one solution) when

(a) one side and the two adjoining angles are given;
(b) two sides and the included angle are given;
(c) three sides are given;
(d) the triangle is a right triangle, and the hypotenuse and a leg are given;
(e) the triangle is a right triangle, and the hypotenuse and an acute angle are given.

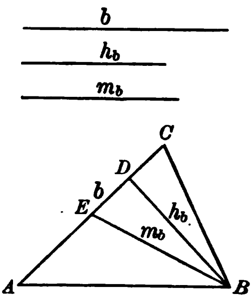
Discuss the limits within which each of these constructions is possible, stating in each case the theorem which defines those limits. Note (§ 182) that two angles of a triangle determine the third. Does this fact make possible an extension of the foregoing list of conditions under which a *triangle* is determined? a *right* triangle?

How many principal parts, and what parts, determine an isosceles triangle? an equilateral triangle?

EXAMPLES

1. Construct the $\triangle ABC$, having given b, h_b, and m_b.

Analysis. Imagine the construction completed, as shown in the figure. Then rt. $\triangle DBE$ is uniquely determined. (Why ?) Also AE and EC are determined. (Why ?) Hence $\triangle ABC$ is uniquely determined.

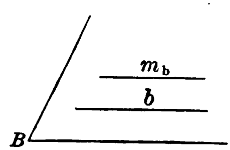


Construction. Construct a rt. $\triangle DBE$, using m_b as the hypotenuse, and h_b as a leg (§ 163). Produce DE indefinitely in both directions. From E lay off on DE produced EA and EC, each equal to one half of b. Draw AB and BC. Proof is unnecessary.

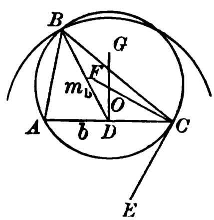
Discussion. The construction is possible only when $m_b > h_b$. Otherwise the values of the parts are not limited.

The principles of loci also are of assistance in determining parts of a figure, when it can be shown that one or more points of the figure are each the intersection of two loci resulting from the conditions of the problem.

2. Construct $\triangle ABC$, given $\angle B$, b, and m_b.



Analysis. Imagine the figure completed. There are not sufficient parts fully to determine any one of the triangles of the figure. But it is apparent that the point B is at the distance m_b from D, the mid-point of AC, and also that it is the vertex of an angle equal to the given $\angle B$, the sides of which pass through A and C.



Hence one locus of the point B is a circle about D as a center, with a radius equal to m_b; and the other locus of the point B is an arc upon AC as a chord, containing $\angle B$ as an inscribed angle (§ 293); and B is determined as either of the two points of intersection of the two loci.

Construction. Lay off $AC = b$. At C construct $\angle ACE = \angle B$. At C draw $CF \perp$ to CE. Bisect AC in D. At D draw $DG \perp$ to AC, intersecting CF in O. About O as a center, with a radius equal to OC, describe a circle. About D as a center, with a radius equal to m_b, describe another circle cutting the first circle in B. Draw BA and BC. $\triangle ABC$ is the triangle required.

(Proof to be completed.)

Discussion. How many solutions are possible ? (Why ?) When is the solution impossible ?

EXERCISES

Triangles

1. Construct a triangle, having given two sides and the angle opposite one of them (a, b, $\angle A$).

Solution. Draw a line AC equal to b. At A construct $\angle CAE$ equal to the given $\angle A$. About C as a center, and with a radius equal to the line a, describe an arc intersecting AE in B.

Discussion. If $\angle A$ is a right angle, how must a compare in length with b? (Why?)

If $\angle A$ is an obtuse angle, how must a compare in length with b? (Why?)

If $\angle A$ is an acute angle, has the side a any lower limit of length? Under what conditions are there two solutions of the problem?

Construct an equilateral triangle, having given:

2. The altitude.

3. The sum of the altitude and one side. (Construct in the figure for analysis a line representing the given sum.)

4. The radius of the circumscribed circle.

5. The radius of the inscribed circle.

Construct a right triangle, having given:

6. A leg and the opposite acute angle.

7. A leg and the altitude on the hypotenuse.

8. A leg and the median on the other leg.

9. An acute angle and the altitude on the hypotenuse.

10. A leg and the bisector of the right angle. (How many degrees in the angle formed by the two given lines?)

11. The hypotenuse and the sum of the legs.

12. The hypotenuse and the difference of the legs.

13. An acute angle and the sum of the legs.

14. One leg and the radius of the inscribed circle.

15. The hypotenuse and the radius of the inscribed circle.

16. The radii of the inscribed and circumscribed circles.

Construct an isosceles triangle, having given:

17. The base and the altitude.

18. The base and the vertex angle.

19. The base and the altitude on one of the legs.

20. The vertex angle and the altitude upon the base.

21. A base angle and the sum of the base and one leg.

Construct $\triangle ABC$ (a, b, c), having given:

22. a, b, h_b.

23. a, b, m_b.

24. a, b, h_c.

25. a, m_a, h_a.

26. $a, h_b, \angle A$.

27. $a, m_a, \angle C$.

28. a, h_a, h_c.

29. $a, h_a, \angle B$.

30. $a, h_a, \angle A$.

31. $h_a, \angle B, \angle C$.

32. $h_a, h_c, \angle C$.

33. a, m_b, c.

34. $a, m_b, \angle C$.

35. $a + b, c, \angle B$.

36. $a + b + c, \angle A, \angle C$.

37. $a + b, c, h_a$.

38. a, m_b, m_c.

39. m_a, m_b, m_c.

40. Mention three groups of three parts each which do not afford a solution of the triangle, and give reasons why they do not.

Quadrilaterals

(Among the "secondary parts" of a quadrilateral are the diagonals; of a parallelogram, the diagonals and the altitudes; of a trapezoid, the altitude and the mid-line.)

41. How many parts are necessary to determine a square? What parts?

42. How many parts are necessary to determine a rhombus? What parts?

43. How many parts are necessary to determine a rectangle? a parallelogram? a trapezoid? a quadrilateral in general?

44. Construct a square, having given the sum of the diagonal and one side.

Construct a rhombus, having given:

45. The diagonals.

46. One angle and one of the diagonals.

47. The base and the altitude.

Construct a rectangle, having given:

48. One side and a diagonal.

49. The perimeter and a diagonal.

50. One side and the corresponding angle between the diagonals.

Construct a trapezoid, having given:

51. The four sides.

52. The bases and the diagonals.

53. The bases and the base angles.

Construct a parallelogram, having given:

54. Two sides and the altitude on one of them.

55. Two diagonals and one side.

56. Two diagonals and the included angle.

Construct an isosceles trapezoid, having given:

57. The bases and one diagonal.

58. The bases and the altitude.

59. The bases and one base angle.

Construct a trapezium, having given:

60. Three sides and the two included angles.

61. The segments of the diagonals made by their point of intersection, and the angle between the diagonals.

Construct a circle, having given:

62. That it has a radius r and is tangent to two intersecting lines.

63. That it has a radius r and is tangent to a given line and also to a given circle.

64. That it touches two given parallel lines and passes through a given point.

65. That it touches a given line at a given point, and passes through a given point without that line.

66. That it has a radius r, passes through a given point, and touches a given circle.

67. Construct the bisector of the angle which two lines would form if produced, without actually producing them to their intersection.

68. In a given triangle, to join two sides by a line of a given length, which shall be parallel to the base.

REVIEW EXERCISES

1. If two intersecting chords make equal angles with the diameter drawn through their point of intersection, the chords are equal.

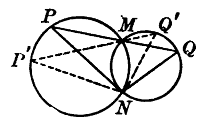
(Let fall ⊥s upon the chords from the center.)

2. The mid-line of a trapezoid circumscribed about a circle is equal to one fourth the perimeter of the trapezoid.

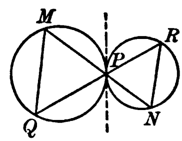
3. If from any point on a circle a chord and a tangent are drawn, the perpendiculars let fall upon them from the mid-point of the intercepted arc are equal.

(Join the mid-point of the arc to the given point on the circle.)

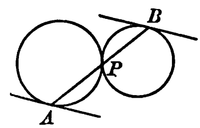
4. Two circles cut each other in two points M and N. Through M the line PQ is drawn meeting the circles in P and Q. Prove that the $\angle PNQ$ is constant for all positions of PQ, so long as it passes through M.



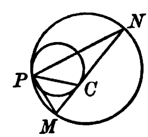
5. Two circles touch each other in P. Through this point of tangency two lines are drawn, one meeting the circles in M and N, the other in Q and R, respectively. Prove that $MQ \parallel NR$.



6. Two circles touch each other in P. AB is a line through P meeting the circles in A and B. Prove that the tangents at A and B are parallel.



7. Two circles touch each other internally at P. MN is a chord of the larger, tangent to the smaller at C. Prove $\angle MPC = \angle CPN$.

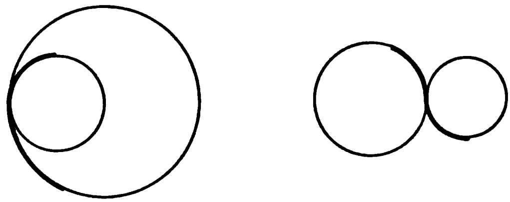


8. If through the points of intersection of two circles parallels are drawn terminating in the circles, these parallels are equal.

9. If two circles are tangent to each other externally, (a) the common internal tangent bisects the common external tangent; (b) the tangents to the circles from any point of the common internal tangent are equal.

State (b) as a locus problem.

10. When two circles are tangent, internally or externally, the point of tangency may be regarded as a transition point from one



circle to the other. Arcs of the two circles which meet at the point of tangency then seem to form one continuous curve. This principle is applied in the construction of moldings, spirals, and other decorative forms, and in laying out curves in railroad building.

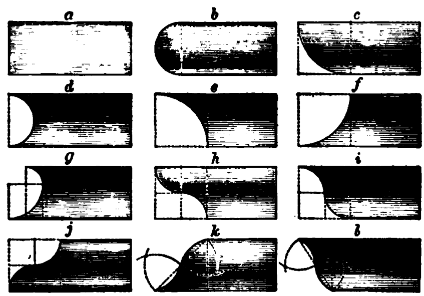


FIG. 1

Copy each of the simple moldings in Fig. 1. In (*g*) the radius of the upper quadrant is one third the height of the molding. In (*k*) and (*l*) the arcs are each 60°.

11. Copy on a larger scale each of the composite moldings in Fig. 2, and explain the construction.

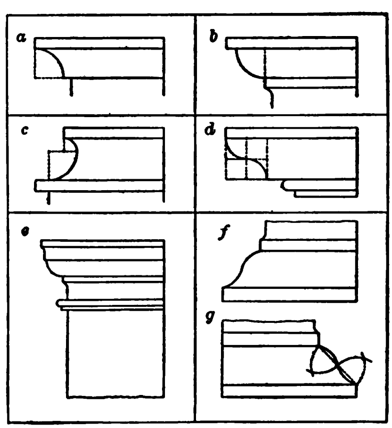
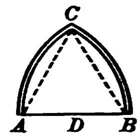


FIG. 2

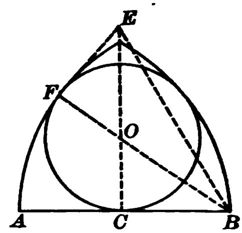
12. Draw a figure illustrating the principle of tangency of circles as shown in the cross section of a ball-bearing wheel and axle. What advantage has this bearing over the ordinary bearing?

13. A form which appears frequently in architectural design is the equilateral Gothic arch, which is formed by using two sides of the equilateral triangle as chords and the third side as a radius, and drawing two arcs, as shown in the figure.



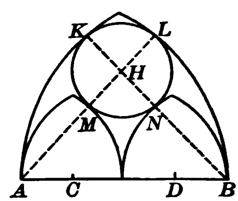
It is required to inscribe a circle in such an arch. Describe a method of construction based on the following analysis:

If O is the center of the required circle, then FB passes through O, and $OC = OF$. (Why?) Whence $\triangle OCB \equiv \triangle OFE$ (a. s. a.) and $EF = CB$. Whence also $\triangle BEC \equiv \triangle BEF$ (rt. △ h. l.). Hence $CE = BF = AB$.

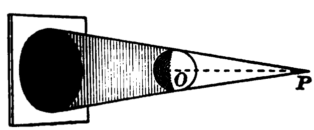


14. In window designing, several equilateral Gothic arches are often combined with one or more circles. A simple combination of this kind is shown in the adjoining figure. Copy this figure and explain its mode of construction.

(The radius to the point of common tangency L passes through H, the center of the circle, and through M, the point of common tangency of the circle and one of the smaller arcs. It follows that AH equals three fourths of AB. Why?)

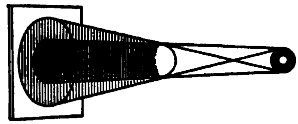


15. An opaque body intercepts the rays of light which shine upon it, leaving a dark space behind. This space is called the *shadow*. The shadow will appear dark in cross section on a screen if the screen is placed beyond the opaque body opposite to the source of light.



When the source of light is a very small body, as, for example, the luminous point of the carbon of an electric arc light, it may be regarded as a geometrical point. In that case the edge of the shadow as shown on the screen is clear and distinct. Suppose that the opaque body is a sphere. Imagine a plane passed through the source of light and the center of the sphere. The boundary of the shadow in this plane is then determined by two lines of indefinite length drawn from the source of light, tangent to the circle.

If the source of light is comparatively large, the shadow will be made up of two portions: the *umbra*, that is, the portion of space from which all light from the source is excluded; and the *penumbra*, which is the portion of space from which part of the light is excluded. Explain how the boundaries of the umbra and penumbra may be determined.



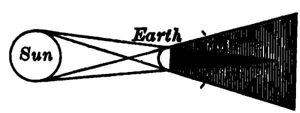
(a) How are the umbra and penumbra affected by a change in the distance apart of the luminous and opaque bodies?

(b) If a croquet ball is held in the path of the sun's rays, is there any penumbra? Explain.

(c) What may be said of the umbra if the luminous sphere and the opaque sphere are of the same size?

(d) What is the shape of the umbra if the luminous sphere is larger than the opaque sphere, as in the case of the sun and the earth?

(e) When the moon passes through the earth's shadow a "lunar eclipse" occurs. The eclipse is "total" or "partial" according as the moon is entirely or partly in the umbra (see the above figure). A solar eclipse arises if the moon's shadow passes across the surface of the earth. Sometimes the earth does not enter the moon's umbra at all on such an occasion. Discuss the relative positions of the sun, the moon, and the earth in that case.



16. If two circles intersect and a line is drawn through each point of intersection terminated by the circumferences, the chords joining the ends of these lines are parallel.

17. Prepare a summary of the properties of two circles in each of the six possible positions.

18. What methods are furnished by the propositions of Book II for proving the equality of line-segments? of angles? of arcs? of chords?

19. Prepare a complete list of all applied problems given in Book II, and also a list of the principles of loci.

LOCUS PROBLEMS

1. In the rectangle $ABCD$ the side AB is twice as long as the side BC. A point E is taken on the side AB, and a circle is drawn through the points C, D, and E. Construct the path of the center of the circle as E moves from A to B.

2. Given a square with each side 3 in. long. Construct the locus of a point P such that the distance from P to the nearest point of the square is 1 in.

3. A straight line 3 in. long moves with its extremities on the perimeter of a square whose sides are 4 in. long. Construct the locus of the middle point of the moving line.

4. A circular basin 16 in. in diameter is full of water, and upon the surface there floats a thin straight stick 1 ft. long. Shade that region of the surface which is inaccessible to the *middle point* of the stick and describe accurately its boundary.

5. The image of a point in a mirror is apparently as far behind the mirror as the point itself is in front. If a mirror revolves about a vertical axis, what will be the locus of the apparent image of a fixed point 1 ft. from the axis?

6. Upon a given base is constructed a triangle, one of whose base angles is double the other. The bisector of the larger base angle meets the opposite side at the point P. Find the locus of P.

7. Find the locus of the middle points of straight lines drawn between two parallel lines.

8. Find the locus of the extremity of a tangent of given length drawn to a given circle.

9. Find the locus of the center of a circle which has a given radius and is tangent to a given circle.

10. Find the locus of the points of contact of tangents drawn from a given point to a given set of concentric circles.

11. Find the locus of the centers of circles which touch two given concentric circles.

12. An angle of 60° moves so that both of its sides touch a fixed circle of radius 5 ft. What is the locus of the vertex?

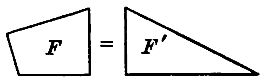
13. If through a fixed point within a circle a chord is drawn, find the locus of the middle point of that chord.

BOOK III

AREA

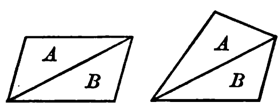
Preliminary Definitions and Exercises

307. Two geometric magnitudes are said to be **equal** if they are of the same size.



Thus, the expression $F = F'$ signifies that the quadrilateral F and the triangle F' occupy equal portions of the plane.

308. It follows at once that *congruent figures are also equal.* Also, if two figures are composed of parts that are respectively congruent, they are evidently equal, though not themselves necessarily congruent.



For example, two congruent triangles may be placed so as to form either a parallelogram or a kite. The two resulting figures are equal, but not congruent.

309. To **transform a figure** means to construct another figure of different form, equal to it.

Thus, we shall learn how to transform any polygon into a triangle, any triangle into a square, etc. The kite in the above figure is a transformation of the parallelogram.

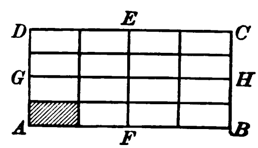
310. To find the **size** of any plane figure, it is necessary to select some **unit of surface,** and to ascertain the number of times this unit is contained in the given figure. A square, constructed on a side of convenient length, is the most satisfactory unit for practical purposes. The standard square units are the square centimeter (sq. cm.), the square meter (sq. m.), the square inch (sq. in.), the square foot (sq. ft.), and other squares, each of whose sides is a standard linear unit.

311. The **area** of a figure is the number showing how many square units of a given kind it contains.

312. The axioms of equality and inequality (§§ 132, 222) apply to areas as well as to line-segments and angles.

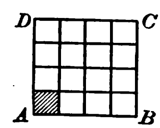
EXERCISES

1. Fold a rectangular sheet of paper $ABCD$ into two equal parts, by placing BC on AD, etc. By repeating this process, divide the given rectangle into small rectangles.



2. Repeat Ex. 1, using a square piece of paper instead of the rectangle. What form will each of the small rectangles take?

3. If in Ex. 1 the small rectangle were taken as the unit of area, what would be the area of the large rectangle?



4. In the second figure the area of the square is said to be 16 square units. What does that mean?

5. Draw a rectangle whose sides are 7 cm. and 5 cm. Divide it into squares. Show that it contains 35 sq. cm.

313. Summary. From the foregoing exercises we conclude:

1. *The area of a rectangle whose sides contain a units and b units of the same kind respectively is* $a \times b$ *square units.*

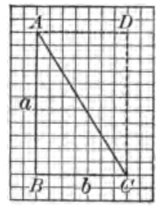
2. *The area of a square whose side contains a units is* a^2 *square units.*

NOTE. The base and the altitude of a rectangle are often called its **dimensions**. In propositions relating to areas the words *rectangle*, *triangle*, etc., are often used for *area of rectangle*, *area of triangle*, etc.

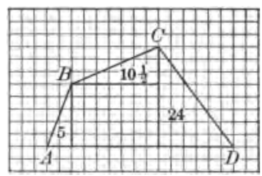
EXERCISES

1. In each of the following exercises plot the points (see § 297), join them by straight lines in the order given, and determine the number of square units in the figure formed:

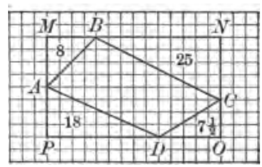
(a) (4, 2), (– 4, 2), (– 4, – 5), (4, – 5).
(b) (3, 2), (– 1, 2), (– 1, – 5), (3, – 5).
(c) (0, 6), (4, 6), (4, – 3), (0, – 3).
(d) (– 2, 3), (– 2, 9), (– 6, 9), (– 6, 3).



2. If a and b represent the number of units in the legs of a right triangle ABC, show that the triangle contains $\frac{a \times b}{2}$ square units.



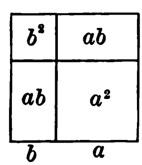
3. The annexed diagrams illustrate how the area of an irregular plotted figure may be found. When one side of the figure coincides with a line of the squared paper, the figure may easily be divided into rectangles and right triangles.



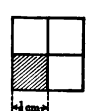
When that is not the case, lines may be drawn through the vertices of the figure, parallel to the ruled lines, thus forming a circumscribed rectangle. By subtracting from that rectangle certain triangles, the required area is obtained.

In each of the following exercises plot the points as in Ex. 1, and find the areas of the plotted figures by the method just outlined:

(a) (1, 1), (10, 1), (7, 7), (3, 5).
(b) (4, 2), (2, 4), (– 2, 4), (– 4, 2), (– 4, – 2), (– 2, – 4), (2, – 4), (4, – 2).
(c) (0, 6), (6, 0), (0, – 6), (– 6, 0).
(d) (0, 3), (4, 0), (6, 4), (2, 7).



4. The accompanying figure illustrates the formula: $(a + b)^2 = a^2 + 2\,ab + b^2$. Explain.



5. Show by a diagram that the square on $\frac{1}{2}$ cm. = $\frac{1}{4}$ sq. cm., that $\left(\frac{1}{3}\text{ cm.}\right)^2 = \frac{1}{9}$ sq. cm., and that $\left(\frac{1}{n}\text{ cm.}\right)^2 = \frac{1}{n^2}$ sq. cm.

6. Construct a square on $2\frac{1}{2}$ cm. Prove geometrically that the area is $6\frac{1}{4}$ sq. cm. Verify numerically.

Suggestion. $(2\frac{1}{2})^2 = (2 + \frac{1}{2})^2 = 4 + 2 + \frac{1}{4} = 6\frac{1}{4}$.

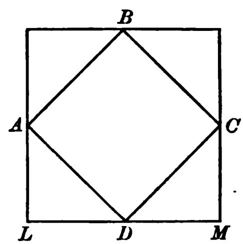
7. Construct a rectangle whose sides are $2\frac{1}{2}$ in. and $1\frac{1}{3}$ in. respectively. Show geometrically that its area is $(2\frac{1}{2} \times 1\frac{1}{3})$ sq. in. $=$ $3\frac{1}{3}$ sq. in.

Suggestion. $2\frac{1}{2} \times 1\frac{1}{3} = \frac{5}{2} \times \frac{4}{3} = \frac{15}{6} \cdot \frac{8}{6} = \frac{120}{36} = 3\frac{1}{3}$.

8. The sides of a rectangle are 2.4 cm. and 3.1 cm. Prove that its area is (2.4×3.1) sq. cm.

Suggestion. 2.4 cm. = 24 mm. Hence, 24 mm. $\times$ 31 mm. = 744 sq. mm. = 7.44 sq. cm. = (2.4×3.1) sq. cm.

9. If the area of the square *ABCD* is 1 sq. in., the outer square must contain 2 sq. in. (Why?) Hence, if the rules developed above are to apply to this square, we must *assume* that its side *LM* is $\sqrt{2}$ in. long. But the value of $\sqrt{2}$ is $1.414\cdots$. This is a decimal which can never be completely written. Does this convince you that there are lines of definite length which we can measure only approximately *in terms of a given unit?* Draw the figure on squared paper.



10. If the dimensions of a rectangle are $\sqrt{2} = 1.414\cdots$, and $\sqrt{3} = 1.732\cdots$, we shall assume that its area is $\sqrt{2} \cdot \sqrt{3} = \sqrt{6} = 2.449\cdots$ sq. ft. Continue the following table and show that the products constantly approach the exact area of the rectangle; that is, $\sqrt{6}$ sq. ft.

$$1.4 \times 1.7 = 2.38.$$
$$1.41 \times 1.73 = \cdots.$$
$$1.414 \times 1.732 = \cdots.$$

11. Find the area of a rectangle if the sides a and b have the values indicated in the following table:

	I	II	III	IV	V	VI	VII
a	7 cm.	.56 in.	$\frac{4}{7}$ ft.	89 mm.	$\sqrt{3}$	$\sqrt{7}$	$x+y$
b	6.5 cm.	.73 in.	6.5 ft.	4.7 cm.	$\sqrt{5}$	$\sqrt{2}$	$x-y$

314. Area of a Rectangle. From the foregoing exercises it appears that *the area of a rectangle in square units is equal to the product of its two dimensions in linear units, even if those dimensions are fractional, decimal, or irrational.*

Ratio

315. If two magnitudes of the same kind, such as two line-segments, contain a certain unit a and b times respectively, then the quotient $\frac{a}{b}$ is often called the **ratio** of these two magnitudes.

For example, the ratio of a line 10 ft. long to one 30 ft. long is $\frac{10}{30}$, or $\frac{1}{3}$. The ratio of m to n is often written $m : n$, or $m \div n$.

In the ratio $a : b$, a and b are called the **terms** of the ratio; a is called the **antecedent** and b the **consequent.**

316. Since ratios are really fractions, their properties are the properties of fractions. Hence

The value of a ratio is not changed if both of its terms are multiplied or divided by the same number.

317. When two ratios $a : b$ and $c : d$ are equal, the four numbers a, b, c, d, are said to be **in proportion** or to be **proportional.** This equality of ratios may be written in any one of the following forms:

$$\frac{a}{b} = \frac{c}{d}, \quad a : b = c : d, \quad a : b :: c : d.$$

These are read "a is to b as c is to d."

318. Since the ratio between two magnitudes of the same kind is obtained exactly or approximately as the quotient of their numerical measures, it is customary to extend the use of the term "ratio" to include the quotient of the numerical measures of two magnitudes of different kinds.

Thus the number of pounds of pressure per square foot of area of a gas inclosed in a vessel is defined as the ratio of the total pressure to the total area of the inclosing vessel.

319. The ratio of two magnitudes of the same kind is independent of the unit by which the magnitudes are measured, since a change in the unit results merely in multiplying or dividing the two terms of the ratio by the same number.

Thus the ratio of the areas of two rectangles is the same, whether they are both measured in square inches or in square centimeters.

320. The ratio of two magnitudes of different kinds is dependent upon the units by which the two magnitudes are measured.

Thus the ratio of the weight of a physical solid in pounds to its volume in cubic feet is not the same as the ratio of its weight in grams to its volume in cubic centimeters.

EXERCISES

1. Simplify the following ratios: $8 : 24$; $3\frac{1}{2} : 7$; $4\frac{1}{3} : 4\frac{1}{6}$; $.4 : .03$; $(a + b)^2 : (a + b)$.

2. What is the ratio of a straight angle to a right angle? of the interior angle of a regular hexagon to the interior angle of an equilateral triangle?

3. Divide an angle of 180° into five parts in the ratio of $1 : 2 : 3 : 4 : 5$. How many degrees in each angle?

4. Divide 50 in the ratio of $m : n$.

5. The sides of a triangle are in the ratio of $3 : 5 : 7$. The perimeter of the triangle is 30. Find the length of each side.

6. Find the ratio of the areas of two rectangles if their respective dimensions in feet are 12 by 27 and 18 by 20; $3\frac{1}{5}$ by $7\frac{1}{2}$ and $2\frac{2}{3}$ by 9.

7. What is the ratio of the area of a given square to the area of the square on its diagonal?

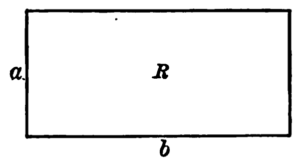
8. What is the ratio of two rectangles, the first of base 10 and altitude x, and the second of base 12 and altitude x?

9. What is the ratio of two rectangles on the same base b, if their respective altitudes are 15 and 20?

10. What is the ratio of the area in square feet of a square, each of whose sides is 6 feet, to the length of one of its sides? Is the ratio the same if the measurements are taken in inches? in yards?

AREAS OF SIMPLE FIGURES

321. Fundamental Principle. *The area of a rectangle is equal to the product of its base and altitude.*



Thus if a and b are the altitude and base respectively of the rectangle whose area is R, then $R = ab$ square units.

322. Corollary 1. *Two rectangles are to each other as the products of their bases and altitudes.*

For if $R = ab$, and $R' = a'b'$, then $\frac{R}{R'} = \frac{ab}{a'b'}$.

323. Corollary 2. *Two rectangles having equal bases are to each other as their altitudes.*

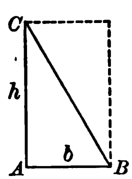
324. Corollary 3. *Two rectangles having equal altitudes are to each other as their bases.*

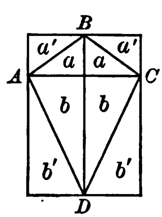
325. Corollary 4. *Two rectangles having equal altitudes and equal bases are equal.*

The above corollaries may be written symbolically as follows:

$$\frac{R}{R'} = \frac{ab}{a'b'}; \ \frac{R}{R'} = \frac{a}{a'}\ [b = b']; \ \frac{R}{R'} = \frac{b}{b'}\ [a = a']; \ R = R' \begin{bmatrix} a = a' \\ b = b' \end{bmatrix}.$$

326. Corollary 5. *The area of a right triangle is equal to one half the product of its legs.*

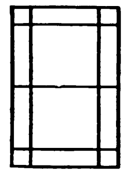




327. Corollary 6. *The area of a kite (rhombus, square, see § 151) is equal to one half the product of its diagonals.*

EXERCISES

1. The edge of a cube is 4 in. How many square feet in its entire exterior surface?

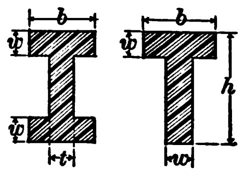


2. A map is drawn so that 1 cm. represents 1000 m. What area is represented by 1 sq. cm.?

3. The dimensions of a rectangular window are 5 ft. and 3 ft. It is to be divided into rectangles and squares as shown in the figure. Determine the dimensions of these parts.

4. The accompanying diagrams represent cross sections of steel beams. Determine the areas of the cross sections, using the dimensions given in millimeters in the following table:

	I	II	III	IV
b	96	72	112	128
w	12	9	14	16
h	192	144	250	300
t	8	6	10	10



5. The diagonals of a square are each 10 ft. long. Find the area of the square (§ 327).

6. The diagonals of a rhombus are 8 ft. and 7 ft. long. Find the area of the rhombus (§ 327).

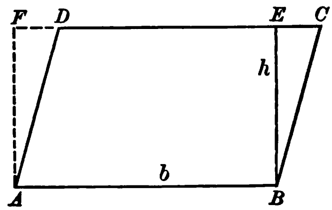
7. If p and A represent the perimeter and the area respectively of a rectangle, determine the dimensions, x and y, in the exercises shown in the following table:

	I	II	III	IV
p	12	16	30	$2a + 2b$
A	8	15	56	ab

8. The Greek historian Thucydides (430 B.C.) estimated the size of the island of Sicily by means of the time it took to sail around it. What was the fallacy in his method?

PROPOSITION I. THEOREM

328. *The area of a parallelogram is equal to the product of its base and altitude.*



Given the parallelogram $ABCD$, with its base AB equal to b, and its altitude BE equal to h.

***To prove** that the area of the* $\square ABCD = b \times h$.

Proof. 1. Draw the line $AF \parallel$ to BE, meeting CD produced at F.

Then $ABEF$ is a rectangle. Why?

2. Also $\square ABCD$ and rectangle $ABEF$ have the same base and the same altitude. Why?

3. But $\triangle ADF \equiv \triangle BCE$. Why?

4. Then area $ABED + \triangle BCE =$ area $ABED + \triangle ADF$. Ax. 2

That is, $\square ABCD =$ rectangle $ABEF$.

5. But area rectangle $ABEF = bh$. Why?

$\therefore$ area $\square ABCD = bh$.

329. COROLLARY 1. *Two parallelograms are to each other as the products of their bases and altitudes.*

330. COROLLARY 2. *Two parallelograms having equal bases are to each other as their altitudes.*

331. COROLLARY 3. *Two parallelograms having equal altitudes are to each other as their bases.*

332. COROLLARY 4. *Two parallelograms having equal bases and equal altitudes are equal.*

[illegible]

[redacted] [illegible]

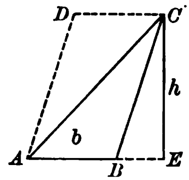
[illegible]

[illegible]

[illegible]

Proposition II. Theorem

333. *The area of a triangle is equal to half the product of its base and altitude.*



Given the triangle *ABC*, with the base *b* and the altitude *h*.

To prove *that the area of the* $\triangle ABC = \frac{1}{2} b \times h$.

Proof. 1. Complete the $\square ABCD$ with AB and BC as consecutive sides.

2. Then the $\square ABCD$ and the $\triangle ABC$ have the same base and altitude.

3. But area $\square ABCD = bh$. § 328

4. Also $\triangle ABC \equiv \triangle ADC = \frac{1}{2} \square ABCD$. Why?

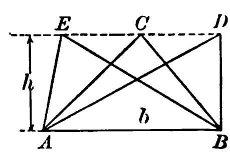
$\therefore$ area $\triangle ABC = \frac{1}{2} bh$.

334. Corollary. *If T and T' are the areas of two triangles with bases b and b', and altitudes h and h', then*

(1) $\frac{T}{T'} = \frac{bh}{b'h'}$, (3) $\frac{T}{T'} = \frac{h}{h'} [b = b']$,

(2) $\frac{T}{T'} = \frac{b}{b'} [h = h']$, (4) $T = T' [h = h',\ b = b']$.

Give the statement in full for each equation.



Discussion. From the proposition it follows that the area of a triangle may be independent of the size of its base angles, as in the case of a parallelogram (Ex. 2, p. 202). Thus, from the accompanying figure, if E, C, and D are points on a line parallel to AB, then $\triangle ABE = \triangle ABC = \triangle ABD$. Explain.

EXERCISES

Problems of Computation

1. Find the area of a triangle if the base b and the altitude h have the values shown in the following table:

	I	II	III	IV	V	VI	VII
b	4	5	7.5	$3\frac{1}{3}$	$x+y$	$a+b$	$\sqrt{3}$
h	7	6	6.5	$4\frac{1}{8}$	$x-y$	$c+d$	$\sqrt{7}$

2. The legs of a right triangle are 4 and 5 (6, 7; $a+b$, $a-b$; x, y). Find the area in each case.

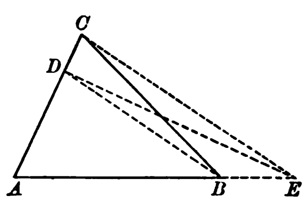
3. The hypotenuse of a right triangle is 10 (20, 30, a, $x+y$). Find the area if the triangle is isosceles.

4. The base of an isosceles triangle is $10\sqrt{3}$; each leg is 10; the vertex angle is 120°. Find the area of the triangle.

5. Each of the triangular faces of the Pyramid of Cheops, in Egypt (p. 1), has a base of nearly 755 ft. and an altitude of nearly 610 ft. How many square feet in the surface of the pyramid?

Problems of Construction

6. Transform $\triangle ABC$ into another triangle having for its base the line AE, where E is any point on AB or AB produced.



Solution. Connect C and E. Draw $BD \parallel$ to EC, and draw DE.

Then $\triangle ADE = \triangle ABC$. Why?

7. Transform a given triangle into another triangle having a base twice as long. What change takes place in the altitude?

8. What is the locus of the vertices of all equal triangles on the same base?

9. Given two equal line-segments. Required to construct on these segments as bases equal triangles having their vertices at the same point. What is the locus of this common vertex?

10. Transform a triangle into an isosceles triangle having the same base.

11. Transform a triangle into another triangle having two sides equal to two lines c and d.

12. Construct a triangle three (four, five, n) times as large as a given triangle.

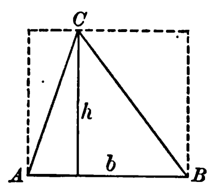
13. Divide a parallelogram into four (six, three, five) equal parts by lines through one vertex.

14. Given the triangle ABC. Extend the three sides in succession, each by its own length. Join the extremities of the segments so constructed. Compare the area of the given triangle with that of each triangle in the figure. What is the result if each side is extended three (four, five) times its own length?

15. Transform a parallelogram into a rhombus.

Theorems

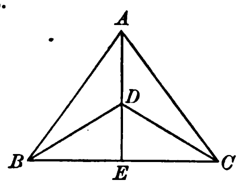
16. Derive a proof for Proposition II from the figure below.



17. A median of a triangle divides it into two equal triangles.

18. The diagonals of a parallelogram divide it into four equal triangles.

19. Any line passing through the intersection point of the diagonals of a parallelogram divides the parallelogram into two equal parts.



20. In the $\triangle ABC$, D is any point in the median AE. Prove that $\triangle ABD = \triangle ADC$.

21. The point in which the three medians of a triangle meet is the vertex of three equal triangles whose bases are the sides of the given triangle.

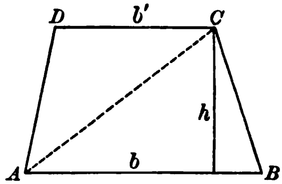
22. The area of an isosceles right triangle is one fourth the square on the hypotenuse.

23. If the mid-points of two sides of a triangle are joined, a triangle is formed which is one fourth the given triangle.

24. The area of a circumscribed polygon is equal to half the product of its perimeter and the radius of the inscribed circle.

Proposition III. Theorem

335. *The area of a trapezoid is equal to half the product of its altitude and the sum of its bases.*



Given the trapezoid *ABCD*, with the bases *b* and *b'* and the altitude *h*.

***To prove** that the area of* $ABCD = \frac{1}{2}h(b + b')$.

1. Draw the diagonal AC.

2. Then $\triangle ABC$, $\triangle ADC$, and trapezoid $ABCD$ have the same altitude.

3. But area $\triangle ABC = \frac{1}{2}bh$, § 333

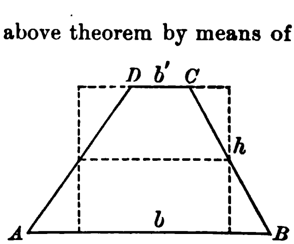
and area $\triangle ADC = \frac{1}{2}b'h$.

4. $\therefore$ area trapezoid $ABCD = \frac{1}{2}h(b + b')$. Ax. 2

336. Corollary. *The area of a trapezoid is equal to the product of its altitude and mid-line.* § 219

EXERCISES

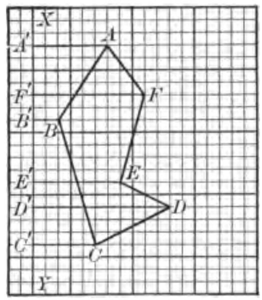
1. From § 321 derive a proof for the above theorem by means of the figure below.



2. In the laying of a railway track an excavation had to be made. A vertical cross section of the excavation perpendicular to the roadbed had the form of a trapezoid with bases of 50 ft. and 68 ft. The depth of the excavation was 12 ft. and its length was 400 ft. Find the area of the cross section. How many cubic yards of dirt had to be removed?

3. The dimensions of a picture are 15 in. and 10 in. The picture is surrounded by a mat 4 in. wide. What is the area of the mat?

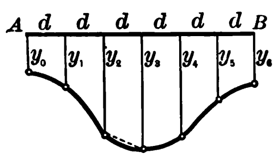
4. The diagram shows how the area of an irregular polygon may be found, if the distance of each vertex from a given **base line,** as XY, is known. These distances AA', BB', etc., are called **offsets,** and are the bases of trapezoids whose altitudes are $A'B'$, $B'C'$, $C'D'$, etc. The area $ABCDEF$ may now be found by the proper additions and subtractions.



On cross-section paper plot the points whose coördinates are given below, join them in order, draw FA, and find the inclosed area in each case:

A	B	C	D	E	F
3, 6	2, 4	3, 0	5, 1	3, 2	4, 5
3, 7	1, 3	4, 0	6, 1	6, 6	5, 7
2, 6	3, 4	1, 2	4, 1	5, 5	3, 7

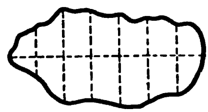
5. In order to determine the flow of water in a certain stream, soundings are taken every 6 ft. on a line AB at right angles to the current. A diagram may then be made to represent a vertical cross section of the stream. If the area of this cross section and the speed of the current are known, it is possible to determine the amount of water flowing through the cross section in a given time.



The required area is often found approximately by joining the extremities of the offsets y_0, y_1, y_2, etc., by straight lines, and finding the sum of the trapezoids thus formed. That is, the strips between successive offsets are replaced by trapezoids. This gives the **Trapezoidal Rule** for finding an area. It may be stated as follows: *To half the sum of the first and last offsets add the sum of all intermediate offsets, and multiply this result by the common distance between the offsets.*

Prove this rule.

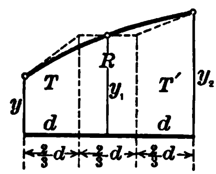
6. Find the area of the cross section of a stream if the soundings taken at intervals of 6 ft. are respectively 4 ft., $5\frac{1}{2}$ ft., 10 ft., $12\frac{1}{2}$ ft., $15\frac{3}{4}$ ft., $8\frac{1}{2}$ ft., and 6 ft.



7. Show how the Trapezoidal Rule may be applied to find approximately the area of a figure bounded by a curved line.

8. In the midship section of a vessel the width taken at intervals of 1 ft. is successively 17, 17.2, 17.4, 17.4, 17.4, 17.2, 16.8, 16, 14, 8, and 2, measurements being in feet. Find the area of the section. (Use the line drawn from the keel $\perp$ to the deck as base line.)

9. A second rule for finding plane areas, known as **Simpson's Rule**, usually gives a closer result than the Trapezoidal Rule. In proving Simpson's Rule two consecutive strips are replaced by a rectangle and two trapezoids as follows: Divide $2\,d$ into three equal parts, erect $\perp$s at the points of division, and complete the rectangle whose altitude is the middle offset y_1, as in the figure. Join the extremities of y and y_2 to the nearer upper vertex of this rectangle. Then if the areas of the trapezoids are T and T', and if the area of the rectangle is R,



$$T = \tfrac{1}{2}(y + y_1) \times \tfrac{2}{3}d, \quad R = y_1 \times \tfrac{2}{3}d, \quad T' = \tfrac{1}{2}(y_1 + y_2) \times \tfrac{2}{3}d.$$

$$\therefore T + R + T' = \tfrac{1}{3}d\,(y + 4\,y_1 + y_2).$$

If, now, the number of strips is *even*, and if the offsets are lettered consecutively $y_0, y_1, y_2, \cdots, y_n$, the addition of the areas of successive double strips, found by the above formula, gives the result

$$\text{Area} = \tfrac{1}{3}d\,(y_0 + 4\,y_1 + 2\,y_2 + 4\,y_3 + \cdots + 2\,y_{n-2} + 4\,y_{n-1} + y_n).$$

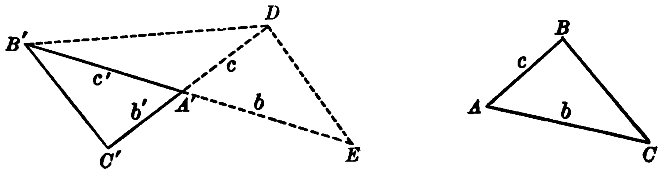
In words: *To the sum of the first and last offsets add twice the sum of all the other even offsets and four times the sum of all the odd offsets, and multiply by one third the common distance between the offsets.*

Solve Exs. 6 and 8 by Simpson's Rule.

10. Find the area of a piece of steel plate which has an axis of symmetry, if the offsets measured on each side from this axis at intervals of 20 cm. are successively 54, 68, 72, 72, 60, 44, 36, 28, and 20, measurements being in centimeters.

Proposition IV. Theorem

337. *If two triangles have an angle of one equal to an angle of the other, their areas are to each other as the products of the sides including the equal angles.*



Given two triangles ABC and $A'B'C'$, having the angle A equal to the angle A'.

To prove that $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{bc}{b'c'}.$$

Proof. 1. Produce $B'A'$ through A' to E, so that $A'E = AC$, and produce $C'A'$ through A' to D, so that $A'D = AB$. Draw DE and $B'D$.

2. Then $\triangle A'DE \equiv \triangle ABC$. s. a. s.

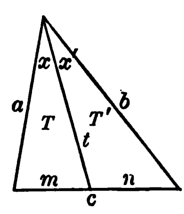
3. Now $\dfrac{\triangle A'DE}{\triangle A'DB'} = \dfrac{b}{c'}$, § 334, (2)

and $\dfrac{\triangle A'DB'}{\triangle A'B'C'} = \dfrac{c}{b'}$. Why?

4. $\therefore \dfrac{\triangle A'DE}{\triangle A'B'C'} = \dfrac{bc}{b'c'}$, Ax. 4

that is, $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{bc}{b'c'}$. Ax. 1

338. Corollary. *The bisector of an interior angle of a triangle divides the opposite side into segments which are to each other as the adjacent sides of the triangle.*



Suggestion. $\dfrac{\triangle T}{\triangle T'} = \dfrac{at}{bt} = \dfrac{a}{b}$. Why?

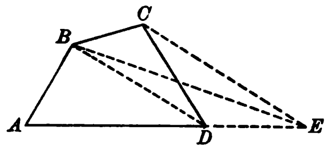
But also $\dfrac{\triangle T}{\triangle T'} = \dfrac{m}{n}$. Why?

This corollary enables us to find the segments determined by the bisector t upon the opposite side.

TRANSFORMATIONS

Proposition V. Problem

339. *To transform a quadrilateral into a triangle.*



Given the quadrilateral *ABCD*.

Required *to construct a triangle equal to $ABCD$.*

Construction. Draw the diagonal BD, and draw $CE \parallel$ to BD, meeting the side AD produced in E. Draw BE. Then BEA is the required triangle.

Proof. 1. $\triangle BCD = \triangle BDE.$ Why ?

2. $\therefore$ quadrilateral $ABCD = \triangle BEA.$ Why ?

340. Corollary. *To transform a pentagon into a triangle,* first transform it into a quadrilateral by the above method, and then transform the quadrilateral into a triangle. A polygon of any number of sides may be transformed into a triangle by a repetition of this process.

EXERCISES

1. In how many ways may a quadrilateral be transformed into a triangle?

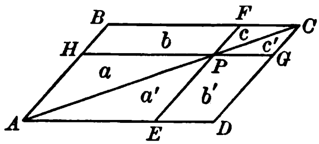
2. Draw on a large scale an irregular hexagon, and transform it into a triangle. Measure the base and the altitude of this triangle, and compute its area. Why is Proposition V of importance?

3. In how many different ways can a pentagon be transformed into a triangle by the above method?

4. Transform a square into a right triangle; into an isosceles triangle.

Proposition VI. Theorem

341. *If through a point on a diagonal of a parallelogram parallels to the sides are drawn, the parallelograms formed on the opposite sides of that diagonal are equal.*



Given the parallelogram *ABCD*, and through *P*, a point on *AC*, the line *EF* parallel to *AB*, and *GH* parallel to *AD*, forming the parallelograms *b* and *b'*, and the triangles *a*, *a'*, *c*, and *c'*.

***To prove** that* $\square b = \square b'$.

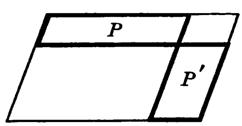
Proof. 1. $\triangle(a + b + c) = \triangle(a' + b' + c')$. Why?

2. But $\triangle a = \triangle a'$ and $\triangle c = \triangle c'$. Why?

3. $\therefore \square b = \square b'$. Why?

This theorem aids in proving some important constructions.

342. Corollary 1. *To transform a parallelogram into another parallelogram whose angles are equal to those of the given parallelogram and whose base is equal to a given line.*



Lay off the new base upon the extension of the original, and complete the figure as shown. Then $P = P'$.

343. Corollary 2. *To transform a triangle into another triangle having a given base.* (Second Method. See p. 204, Ex. 6.)

Since any triangle may be regarded as one half of a parallelogram, the above construction may be extended to include triangles.

EXERCISES

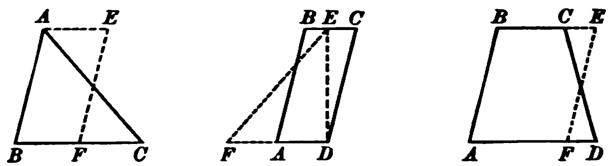
Simple Transformations

1. Transform a $\triangle ABC$ into a triangle with its base on BC, or BC extended, and with the opposite vertex (a) on AC; (b) on AB extended; (c) within $\triangle ABC$; (d) without $\triangle ABC$. (See Ex. 6, p. 204.)

2. Transform a rectangle into (a) another rectangle having a different base; (b) a parallelogram of given base and given angles; (c) a parallelogram of given sides; (d) a triangle of given base; (e) a trapezoid of the same altitude and a given base.

3. Transform a parallelogram into (a) a rectangle of given altitude; (b) a parallelogram of given altitude; (c) a triangle of given altitude; (d) a trapezoid of the same altitude.

4. The figures show easy methods of transforming (a) a triangle into a parallelogram; (b) a parallelogram into a triangle; (c) a trapezoid into a parallelogram. Explain.



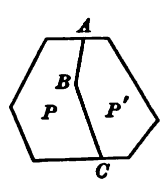
Divisions and Dissections

5. From one vertex of a triangle draw lines dividing the triangle into (a) three (five, six) equal parts; (b) two parts that shall be in the ratio of $3:1$ ($4:1$; $3:4$); (c) three parts that shall be in the ratio of $1:2:3$ ($2:3:4$).

6. Solve Ex. 5 if a parallelogram is given instead of a triangle.

7. Divide a trapezoid into two (three, five) equal trapezoids.

Suggestion. Divide each of the bases into as many equal parts as the trapezoid is to have.

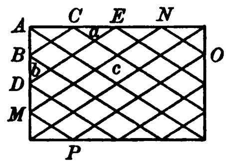


8. A polygon is divided into two parts, P and P', by the broken line ABC. Explain how ABC may be replaced by a straight line so as not to increase the area of either P or P'.

9. Divide a quadrilateral into three equal parts by drawing lines from one vertex.

Suggestion. Transform the quadrilateral into a triangle having that vertex as its vertex, and proceed as in Ex. 5.

10. Show at least six ways of dividing a given parallelogram into four equal parts.



11. In the accompanying rectangular figure, representing an ornamental window, what part of the total area is $\triangle a$? $\triangle b$? $\square c$? $\triangle ABC$? $\triangle ADE$? $\square MNOP$?

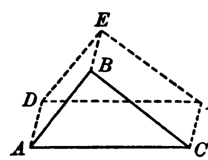
12. Transform into a triangle, and then into a rectangle, (a) a pentagon; (b) an octagon.

Addition of Triangles and Parallelograms

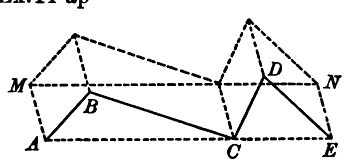
13. Construct a triangle equal to the sum of two given triangles.

Suggestion. Transform the first of the given triangles into another triangle having its base equal to the base of the other given triangle.

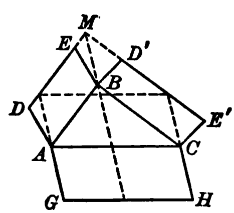
14. Given the triangle ABC. If a line-segment equal to AD moves along AB and BC, remaining constantly parallel to AD, prove that $\square AE + \square EC = \square AF$. (Prove $\triangle ABC \equiv \triangle DEF$ by s. s. s.)



15. Show that the principle of Ex. 14 applies to a broken line $ABCDE$.



16. Theorem due to Pappus (300 A.D.). If on two sides of a triangle ABC any desired parallelograms AE and BE' are constructed, and the sides DE and $D'E'$ are produced to meet at M, and on AC a parallelogram is constructed having a side AG equal and parallel to BM, then $\square AH = \square AE + \square BE'$. (Cf. Ex. 14.)

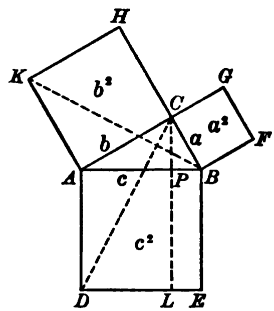


17. Construct a parallelogram equal to the sum of two given parallelograms.

THEOREM OF PYTHAGORAS

Proposition VII. Theorem

344. *In a right triangle the square on the hypotenuse is equal to the sum of the squares on the legs.*



Given the right triangle *ABC*, having the legs *a* and *b* and the hypotenuse *c*, with the square *AE* on the hypotenuse *AB*, and the squares *CF* and *CK* on the legs.

To prove that $c^2 = a^2 + b^2$.

Proof. 1. Draw $CL \parallel$ to AD. Draw BK and CD.

2. Now ACG and BCH are straight lines. Why?

3. Also $\triangle ACD \equiv \triangle AKB$. s. a. s.

For $AC = AK$, Why?

$AB = AD$,

and $\angle CAD = \angle KAB$. Why?

4. But area rect. $AL = 2$ area $\triangle ACD$, Why?

and area square $CK = 2$ area $\triangle AKB$.

5. $\therefore$ rect. $AL =$ square CK. Ax. 1

6. In like manner, by drawing CE and AF, it may be proved that

$$\text{rect. } BL = \text{square } CF.$$

7. But square AE = rect. AL + rect. BL. Ax. 7

$\therefore$ square AE = square CK + square CF. Ax. 1

That is, $c^2 = a^2 + b^2$. Why?

345. Corollary. *The square on one leg of a right triangle is equal to the square on the hypotenuse diminished by the square on the other leg.*

346. Historical Note. Proposition VII is called the Pythagorean theorem, as its discovery is usually credited to Pythagoras (about 550 B.C.). It is practically certain, however, that the theorem was known to the Egyptians and the Hindus long before that time.

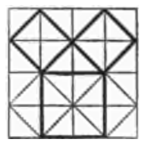


Fig. 1

The Pythagorean proposition is in many respects the most famous single truth of plane geometry. It is of the greatest importance, and many different proofs for it have been given. The one reproduced above is due to Euclid (300 B.C.).

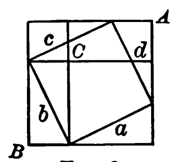


Fig. 2

347. Alternative Proofs. Fig. 1 shows how the proof might have been suggested in the laying of tile floors.

Fig. 2 is thought by some writers to have been used by Pythagoras. If $\triangle$ a, b, c, d, are taken from the large square, the square on the hypotenuse remains, and if the rectangles AC and CB are removed, the squares on the legs remain. But the two rectangles are together equal to the four triangles. Hence the remainders are equal.

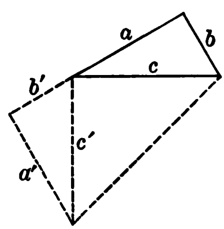


Fig. 3

348. Fig. 3 is used in a proof ascribed to President Garfield. On the hypotenuse construct one half of a square, and draw a' perpendicular to a produced. Prove that $\triangle abc \equiv \triangle a'b'c'$, and that the entire figure is a trapezoid of altitude $a + b'$, etc.

EXERCISES

Numerical Problems

(A knowledge of how to extract the square root of a number is necessary for the following problems. Results should be obtained correct to two decimal places, unless otherwise specified. If it is required to calculate the length of one leg a of a right triangle when the other leg b and the hypotenuse c are given, it will be found helpful to bear in mind that the equation $a^2 = c^2 - b^2$ may be written $a^2 = (c + b)(c - b)$. For example, if $c = 37$, $b = 35$, then $a^2 = 37^2 - 35^2 = (37 + 35)(37 - 35) = 72 \times 2 = 144$. $\therefore a = 12$.)

1. If x and y in the following table represent the lengths of the legs of a right triangle, find the length of the hypotenuse in each case:

x	4	12	24	a	$a+b$
y	3	5	7	b	$c+d$

2. If, in Ex. 1, x represents the hypotenuse and y one of the legs, find the other leg.

3. Find the diagonal of a rectangle whose dimensions are 8 and 15; 11 and 60; 13 and 84.

4. A parallelogram is inscribed in a circle whose diameter is 13. One side of the parallelogram is 5. Find the other side.

5. The base of an isosceles triangle is 4, and its legs are each 5. Find the altitude and the area.

6. The side of an equilateral triangle is 6. Find the area.

7. The bases of an isosceles trapezoid are 6 and 12, and the legs are each 5. Find the area.

8. The diagonals of a rhombus are 6 and 8. Find its perimeter.

9. A chord 48 in. long is 7 in. from the center of its circle. How long is the radius?

10. If the coördinates of two points, P and Q, are (2, 1) and (6, 4) respectively, find the length of the line PQ.

11. If the coördinates of the vertices of a triangle are (1, 1), (3, 3), and (1, 5), find the perimeter of the triangle; find its area.

12. Pythagorean Numbers. Prove that the three integers in each of the following groups may represent the sides of a right triangle.

(a) 3, 4, 5.	(d) 8, 15, 17.
(b) 5, 12, 13.	(e) 11, 60, 61.
(c) 7, 24, 25.	(f) 12, 35, 37.

NOTE. The above list may be extended by the use of the following formulas:

(1) If the length of a leg is denoted by an *even* integer n, the lengths of the other two sides are $\frac{n^2-4}{4}$ and $\frac{n^2+4}{4}$. (Plato.)

(2) If n is *odd*, the other two integers are $\frac{n^2-1}{2}$ and $\frac{n^2+1}{2}$.

(Pythagoras.)

13. The centers of two circles are 25 in. apart. Their radii are 3 in. and 10 in. respectively. Find the length of their common external tangent (see p. 161).

14. The centers of two circles are 13 in. apart. If their radii are 3 in. and 2 in. respectively, what is the length of the common internal tangents? (See p. 161.)

APPLIED PROBLEMS

1. What is the diagonal of a rectangular floor whose dimensions are 12 ft. and 9 ft.?

2. Find how far a pedestrian is from his starting point if he walks (a) 12 mi. N. and then 5 mi. E.; (b) 20 mi. S.E. and then 21 mi. S.W.; (c) 5 mi. N., 3 mi. E., 2 mi. N.; (d) 2 mi. W. and then 10 mi. S.W.

3. A captive balloon rises vertically to a height of 1000 ft. How far is it from an observer 400 ft. away from the point of ascent?

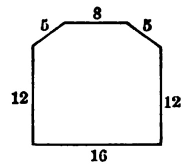
4. A ladder 25 ft. long reaches to a window 24 ft. from the ground. How far is the foot of the ladder from the wall?

5. Two telephone poles are 60 ft. apart. The height of the poles is 40 ft. and 30 ft. How long is a wire connecting their tops?

6. A ladder 50 ft. long is placed against a window 40 ft. from the ground. The upper extremity of the ladder slides down 1 ft. How far does the other extremity of the ladder slip?

7. A derrick has a movable arm 41 ft. long. A weight to be lifted is 9 ft. from the foot of the arm. How long is the cable extending from the end of the arm to the weight?

8. The side of a square baseball field is 90 ft. Find the distance from second base to the home plate.



9. A hexagonal floor has the form and the dimensions indicated in the figure. Find the area.

10. An elevated train moving at the rate of 25 mi. an hour passes above a surface car going at the rate of 15 mi. an hour. The two tracks cross at right angles. Find the distance separating the train and the car after 10 min.

11. A coast-defense gun has a range of 10 mi. A boat sails along the shore at a distance of 8 mi., going at the rate of 18 mi. per hour. How long is the boat within the range of the gun?

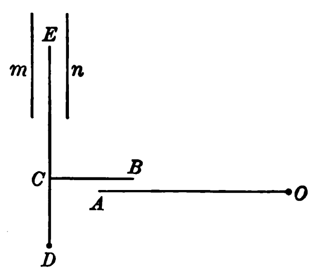
12. A pendulum is 39.1 in. long. Its bob is drawn to one side until it is 1 in. higher than it was originally. How far is it from the vertical line passing through the point of support of the pendulum?

13. A teeter board 12 ft. long is supported at its center by a frame 3 ft. high. How high can each extremity of the board rise?

14. A box has the form of a cubic meter. How many millimeters in the diagonal of the box?

15. The dimensions of a rectangular room are 16 ft., 12 ft., and 9 ft. How long is a line connecting a lower corner with the opposite upper corner?

16. The diagram represents a lever OA, 2 ft. long, which may revolve about O in a vertical plane. In the plane of its revolution it is made to push against a horizontal rod BC attached to a vertical rod DE. B projects 4 in. beyond A. DE is compelled to move vertically by the guides m and n. As OA revolves, how high is DE raised by it?



17. A tree is broken 24 ft. from the ground. The two parts hold together and the top of the tree touches the ground 7 ft. from the foot. Find the height of the tree.

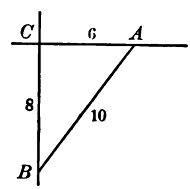
18. In the middle of a pond 10 ft. square grew a reed. The reed projected 1 ft. above the surface of the water. When blown aside by the wind, its top part reached to the mid-point of a side of the pond. How deep was the pond? (Old Chinese problem.)

19. The width of a house is 28 ft. The triangle in the gable is right-angled. How long is each of the rafters in this triangle? What is the area of the triangle? How high is it?

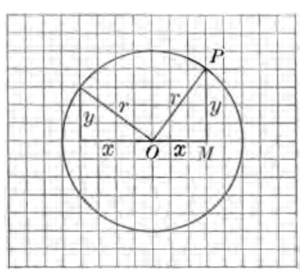
20. A house is 32 ft. wide, 45 ft. long, 25 ft. high to the roof and 35 ft. high to the ridgepole. How many square feet are there in its entire exterior surface?

21. A circular pavilion has a conical roof. The pavilion is 20 ft. high to the roof, and 30 ft. high to the vertex of the cone. The diameter of the pavilion is 25 ft. The flagpole is 12 ft. high. How long is the guy rope joining the top of the flagpole to the edge of the roof?

22. Masons and carpenters, in laying out the plan of a rectangular building, often use the following method of constructing the right angles at the corners: Three poles, whose lengths are 6 ft., 8 ft., and 10 ft. respectively, are joined together at their extremities so as to form a triangle. The angle opposite the largest pole is a right angle. (Why?) A similar method is known to have been used by the surveyors, or "rope-stretchers," of ancient Egypt.



23. The theorem of Pythagoras is employed to find the "equation of a circle" about the origin as a center.

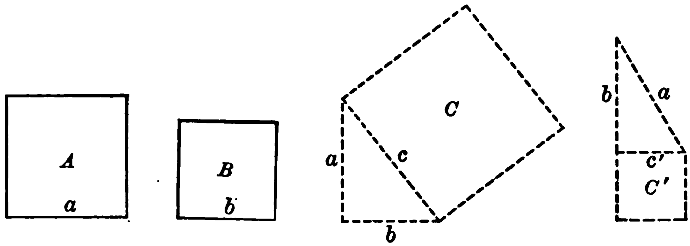


Take any point P in a circle about the origin O. Draw the ordinate MP. Let $OM = x$, and $MP = y$. Then $\overline{OM}^2 + \overline{MP}^2 = \overline{OP}^2$. If the radius $OP = r$, this becomes $x^2 + y^2 = r^2$. This equation holds for the coördinates of any point on the circle, and is called the *equation of the circle*, r being any known number.

24. Plot the equations $x^2 + y^2 = 25$; $x^2 + y^2 = 98$. (See Ex. 23.)

Proposition VIII. Problem

349. *To construct a square equal to the sum or the difference of two given squares.*



Given two squares *A* and *B*.

***Required** to construct a square C equal to A + B, and a square C' equal to A − B.*

(To be completed.)

EXERCISES

1. Construct a square equal to the sum of three given squares.

2. If a, b, and c are line segments, construct

$$x = \sqrt{a^2 + b^2};\ x = \sqrt{a^2 + b^2 + c^2}.$$

3. Construct a square twice as large as a given square; three times as large.

4. If squares be constructed on the diagonals of a rectangle, prove that their sum equals the sum of the squares constructed on the sides of the rectangle.

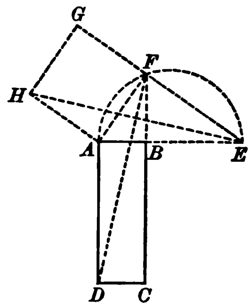
5. Represent by squares the areas of the following countries and states, using a scale in which 1 cm. represents 100 mi.:

United States (including Alaska) . . .	3,605,600 sq. mi.
Texas	265,780 sq. mi.
California	158,360 sq. mi.
New York	49,170 sq. mi.

6. Represent by three squares the area of the state in which you live and of two neighboring states. Then construct a square representing the combined areas of the three states.

PROPOSITION IX. PROBLEM

350. *To construct a square equal to a given rectangle (or parallelogram).*



Given the rectangle *ABCD*.

Required *to construct a square equal to $ABCD$.*

Construction. 1. Produce AB, the shorter side of $ABCD$, to E, making $AE = AD$, and on AE as a diameter construct a semicircle.

2. Construct $BF \perp$ to AE, meeting the semicircle at F.

3. Draw AF. The square on AF is the required square.

Proof. 1.	Draw HE and DF.	
2. Then	$\triangle ADF \equiv \triangle AHE$.	s. a. s.
3. But	area rect. $AC = 2$ area $\triangle ADF$,	Why?
and	area square $FH = 2$ area $\triangle AHE$.	
4. Hence	rect. $ABCD =$ square HF.	Ax. 1

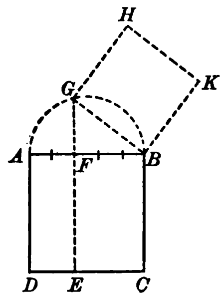
351. COROLLARY 1. *To transform a triangle into a square,* first transform the triangle into a rectangle having the same base as the triangle and an altitude equal to one half the altitude of the triangle, and then transform that rectangle into a square.

352. COROLLARY 2. *To transform a polygon into a square,* first transform the polygon into a triangle (§ 340), and then transform that triangle into a square, as above.

NOTE. Proposition IX enables us to find the area of any polygon by one measurement, namely, that of the side of the equal square.

Proposition X. Problem

353. *To construct a square which shall be a given part of a given square.*



Given the square *ABCD*.

***Required** to construct a square equal to $\frac{3}{5}$ of $ABCD$.*

Construction. 1. On AB construct $BF = \frac{3}{5} AB$. § 212

2. Draw $EF \parallel$ to AD.

3. Transform the rectangle $EFBC$ into a square BH (§ 350). This is the required square.

(Proof to be completed.)

EXERCISES

Constructions

1. Construct a square equal to a triangle of base 7 cm. and altitude 5 cm.

2. Construct $\sqrt{21}$.

Solution. Let $\sqrt{21} = x$. Then $x^2 = 21 = 7 \cdot 3$. (Apply § 350.)

3. Construct $\sqrt{12}$ in. by several methods. Compare your final results.

4. Construct $x^2 = \frac{3}{4} a^2$. (*Suggestion.* $x^2 = \frac{3}{4} a \cdot a$.)

5. Transform a pentagon into a square.

6. Transform an isosceles triangle into a square.

7. Construct a square equal to $\frac{1}{5}$ of a given hexagon.

8. Construct a square equal to the sum of two given triangles.

NUMERICAL PROBLEMS

1. If the side of a square is a and its diagonal is d, show that $d = a\sqrt{2}$ and that $a = \frac{d}{\sqrt{2}}$.

2. The diagonal of a square is 28 cm. Find the length of a side.

Solution. $a = \frac{28}{\sqrt{2}} = \frac{28}{1.4} = \frac{28}{\frac{14}{10}} = \frac{280}{14} = 20$, approximately.

3. If the hypotenuse of an isosceles right triangle is c, show that the area of the triangle is $\frac{c^2}{4}$.

4. The length of the leg of an isosceles right triangle is 7. What is the area? What is the hypotenuse?

5. The hypotenuse of an isosceles right triangle is 10. What is the area? How long is each leg?

6. If a side of an equilateral $\triangle$ is a, show that the area is $\frac{a^2}{4}\sqrt{3}$.

7. Find the area of an equilateral $\triangle$ in terms of its altitude.

8. The side of an equilateral triangle is 6. Find the altitude and the area.

9. A side of a regular hexagon is 8. What is the area of the hexagon?

10. A regular hexagonal prism 10 in. high has base edges of 4 in. How many square inches in the entire exterior surface?

11. The base edges of a pyramid with a square base are 5 in. in length, the height of the pyramid being 7 in. Find the length of the other edges.

12. A triangular pyramid is formed by 4 congruent equilateral triangles of side 10 cm. Such a pyramid is called a **regular tetrahedron.** How many square centimeters in its entire surface?

13. A right triangle has legs of 6 cm. and 8 cm. Find the altitude on the hypotenuse.

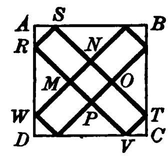
14. A rhombus has diagonals of 12 in. and 18 in. Find its altitude.

15. A rhombus whose side is 5 in. has acute angles of 60°. What is its area?

16. What is the result in Ex. 15, if the acute angles are 45°?

17. A square frame consists of four sticks hinged together at their extremities. The length of each stick is 8 in. The frame is drawn to one side until two of the angles become 120°. Is the area increased or decreased, and how much?

18. A square frame, 20 in. on a side, is to be stiffened by a diagonal crosspiece. How long is the crosspiece?



19. The side of a square is 5 cm. Equal distances of 1 cm. are laid off on the sides from the vertices. Find the length of RS; of ST; of MN. Find the area of $RSTV$; of $MNOP$; of RMW.

20. A staircase has 11 stairs, each $2a$ in. wide and a in. high. How long a board will completely protect it?

21. The edge of a cube is 1 in. Find its diagonal.

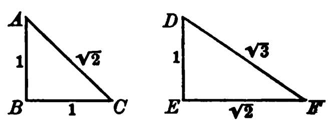
22. The dimensions of a rectangular room are a, b, and c. How long is a diagonal from a lower corner to the opposite upper corner?

Square Roots

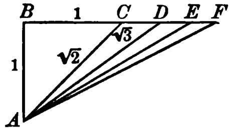
23. If $AB = 1$, and $BC = 1$, AC is a geometric representation of $\sqrt{2}$. Prove.

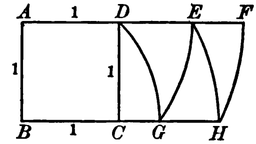
24. If $DE = 1$, and $EF = \sqrt{2}$, prove that $DF = \sqrt{3}$.

25. Construct $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, using squared paper.



26. The following figures illustrate two short methods of representing square roots of small integers. Explain.



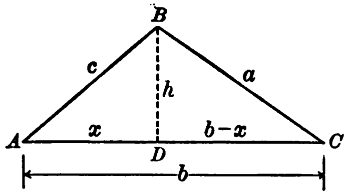


$AB = BC = 1.$
$AC = BD = \sqrt{2}.$
$AD = BE = \sqrt{3}$, etc.

AC is a square.
$BD = BG = \sqrt{2}.$
$AG = AE = \sqrt{3}$, etc.

PROPOSITION XI. THEOREM

354. *If a, b, and c denote the sides of a triangle, and* $s = \frac{1}{2}(a+b+c)$, *the area of the triangle is* $\sqrt{s(s-a)(s-b)(s-c)}$.



Given the triangle *ABC*, the angle *A* being acute.

To prove that the area of $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}$.

Proof. 1. Draw the altitude BD, and denote it by h.

2. The area of

$\triangle ABC = \frac{1}{2} b \cdot h.$ § 333

3. Now $h^2 = c^2 - x^2 = a^2 - (b-x)^2.$ Why?

4. $\therefore x = \dfrac{c^2 + b^2 - a^2}{2b}.$ Why?

5. But $h^2 = c^2 - x^2 = (c+x)(c-x).$ Why?

$$\therefore h^2 = \left(c + \frac{c^2+b^2-a^2}{2b}\right)\left(c - \frac{c^2+b^2-a^2}{2b}\right)$$
$$= \frac{(2bc + c^2 + b^2 - a^2)(2bc - c^2 - b^2 + a^2)}{4b^2}$$
$$= \frac{(b+c+a)(b+c-a)(a-b+c)(a+b-c)}{4b^2}.$$

6. If $a + b + c = 2s$, then $b + c - a = 2s - 2a = 2(s-a)$, etc.

7. $\therefore h^2 = \dfrac{2s \cdot 2(s-a) \cdot 2(s-b) \cdot 2(s-c)}{4b^2}.$

8. $\therefore h = \dfrac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}.$

9. Hence the area of

$$\triangle ABC = \tfrac{1}{2} b \cdot h$$
$$= \frac{1}{2} b \cdot \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

Historical Note. This is known as **Hero's formula** for the area of a triangle. Hero of Alexandria (about 75 A.D.) was a famous Greek surveyor.

EXERCISES

1. The sides of a triangle are 10, 17, 21. Find its area.

Solution. Since $s = 24$, $A = \sqrt{24 \cdot 14 \cdot 7 \cdot 3}$
$= \sqrt{4 \cdot 6 \cdot 7 \cdot 2 \cdot 7 \cdot 3}$
$= \sqrt{4 \cdot 6^2 \cdot 7^2} = 84.$

2. The sides of a triangle are (a) 26, 35, 51; (b) 75, 176, 229; (c) 104, 111, 175. Find the area in each case.

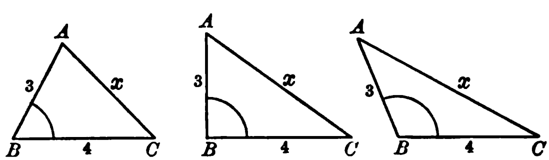
3. Explain how the area of any polygon may be found by Proposition XI.

4. Given the area A of a triangle whose sides are a, b, c. The altitudes are denoted by h_a, h_b, h_c respectively. Prove that $h_a = \frac{2A}{a}$, $h_b = \frac{2A}{b}$, $h_c = \frac{2A}{c}$. Find the altitudes of the $\triangle$ in Ex. 1 and Ex. 2.

5. A triangular park is bounded on all sides by streets. Its sides are 208 ft., 222 ft., and 350 ft. How far is it from each corner to the opposite street? Which of these distances is the shortest?

6. If two rods AB and BC, 3 and 4 units long respectively, are hinged together at B, and AB revolves about B, the length of AC (denoted by x) evidently depends on the size of $\angle B$.

1. If $\angle B = 90°$, $x^2 = 3^2 + 4^2$. Hence
2. If $\angle B < 90°$, $x^2 = 3^2 + 4^2$ minus some quantity.
3. If $\angle B > 90°$ and $< 180°$, $x^2 = 3^2 + 4^2$ plus some quantity.



Explain how these relations enable us to find out whether a triangle is acute, right, or obtuse, when we know its sides (§§ 231, 232).

7. The sides of a triangle are 6, 7, 8. What kind of triangle is it?

8. Given four rods of lengths 4, 5, 6, 8. If they are hinged together at their extremities, three at a time, how many different triangles can be made? What kind of triangle results in each case?

9. The base of the gable triangle of a roof is 28 ft. Each of the oblique rafters is 17 ft. long. What kind of triangle is it?

IMPORTANT FORMULAS

Express in words the following formulas:

Rectangle,	$A = b \times a.$
Square,	$A = b^2.$
Kite,	$A = \frac{d \times d'}{2}.$
Parallelogram,	$A = b \times h.$
Triangle,	$A = \frac{1}{2} b \times h$ $= \sqrt{s(s-a)(s-b)(s-c)}.$
Equilateral triangle,	$A = \frac{a^2}{4}\sqrt{3}.$
Trapezoid,	$A = \frac{1}{2} h(b + b').$
Circumscribed polygon,	$A = \frac{1}{2} p \times r.$
Square,	$d = a\sqrt{2};\ a = \frac{d}{\sqrt{2}}.$
Triangle,	$h_b = \frac{2A}{b}.$
Equilateral triangle,	$h = \frac{a}{2}\sqrt{3}.$
Right triangle,	$h_c = \frac{a \times b}{c}.$

REVIEW EXERCISES

1. Why is a square the most convenient unit of area?

2. Why is the Pythagorean theorem so important?

3. In how many ways can you find the area of a polygon?

4. What is the importance of Proposition V?

5. The dimensions of a rectangle are 4 in. and 5 in. What is the altitude of an equal equilateral triangle?

6. The diagonals of a rhombus are 8 and 7. What is the side of an equal square?

22. Verify the formulas given in engineering works for the following shapes:

In Fig. 1, area $= (d + 2c)t$. In Fig. 2, area $= dt + (s + y)2z$.

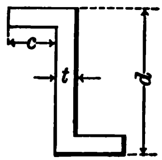


FIG. 1

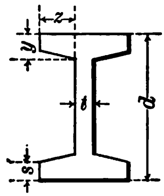


FIG. 2

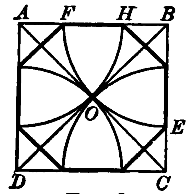


FIG. 3

23. In Fig. 3 is shown a rapid method of constructing a regular octagon. $ABCD$ is a square. Use the vertices as centers, and distances equal to AO as radii for the interior arcs. Prove that the octagon is equiangular and equilateral.

Suggestion. Let $AB = 2a$. Then $AO = a\sqrt{2}$, and $AF = 2a - a\sqrt{2}$.

24. If one side of a regular octagon is a, prove that its area is $2a^2(\sqrt{2} + 1)$.

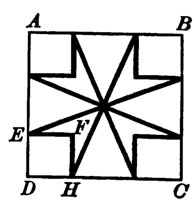
Suggestion. Divide the octagon into rectangles and triangles.

25. What is the area of the octagon in Ex. 23, in terms of AB?

26. One of the largest chimneys in the United States rests on a concrete octagonal foundation 40 ft. wide and 23 ft. deep. How many cubic yards of concrete were used in the construction of the foundation? (See Exs. 23–25.)

27. A flagpole has a concrete octagonal foundation, each side of the octagon being $2\frac{1}{2}$ ft. long. How large an area does the foundation cover?

28. A table top has the form of a regular octagon, each side being 1 ft. long. Find its area.



29. The polishing drums in a button factory have the form of prisms with regular octagons for bases. If each side of the octagonal base is 10 in. and the length of the drum is 3 ft., what is the capacity of each drum?

30. The construction of the above figure is apparent. AC and EH are squares. If $AB = 4a$, and $DE = a$, find the area of the cross.

BOOK IV

PROPORTIONAL MAGNITUDES. SIMILAR POLYGONS

355. Extremes and Means. Fourth Proportional. In the proportion

$$a : b = c : d,$$

the terms a and d are called the **extremes**, and the terms b and c are called the **means**.

The **fourth proportional** to three given numbers is the fourth term of the proportion which has for its first three terms the three given numbers taken in order.

Thus in the proportion $4 : 8 = 5 : 10$, 10 is the fourth proportional to 4, 8, and 5.

356. Continued Proportion. The numbers a, b, c, d, are said to be in **continued proportion** if $a : b = b : c = c : d$. If three numbers are in continued proportion, the second is called the **mean proportional** between the other two, and the third is called the **third proportional** to the other two.

Thus in the proportion $2 : 4 = 4 : 8$, 4 is the mean proportional between 2 and 8, and 8 is the third proportional to 2 and 4.

357. Definition of Inversely Proportional. The terms of one ratio are said to be **inversely proportional** to the terms of another when the first ratio is equal to the reciprocal of the second ratio.

Thus a and b are said to be inversely proportional to x and y, if $a : b = y : x$. For example, the two altitudes of a parallelogram are inversely proportional to the corresponding bases. If the sides are a and b and the corresponding altitudes h_a and h_b, then $ah_a = bh_b$. (Why?) Whence, dividing both members by the product ah_b,

$$h_a : h_b = b : a.$$

That is, the altitudes h_a and h_b of the parallelogram are inversely proportional to the corresponding bases a and b.

FUNDAMENTAL PRINCIPLES

358. Theorem I. *In every proportion the product of the extremes is equal to the product of the means.*

For if $\frac{a}{b} = \frac{c}{d}$, we may multiply the equals by bd. Then $ad = bc$. This principle enables us to find any term of a proportion, if the other three are given.

359. Theorem II. *The mean proportional between two numbers is equal to the square root of their product.*

For if $a : b = b : c$, then $b^2 = ac$. Therefore $b = \sqrt{ac}$.

360. Theorem III. *If the product of a pair of numbers equals the product of a second pair, the four numbers will be in proportion when written in any order that makes one pair the extremes and the other pair the means.*

Thus if $ad = bc$, then it readily follows that

1. $a : b = c : d$.	5. $b : a = d : c$.
2. $a : c = b : d$.	6. $b : d = a : c$.
3. $d : b = c : a$.	7. $c : a = d : b$.
4. $d : c = b : a$.	8. $c : d = a : b$.

361. Theorem IV. *In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Proof. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$, let x represent the value of each of the ratios.

Then $$a = bx,\ c = dx,\ e = fx,\ g = hx.$$

Hence $$a + c + e + g = (b + d + f + h)\,x. \qquad \text{Why?}$$

Then $$\frac{a + c + e + g}{b + d + f + h} = x = \frac{a}{b} = \frac{c}{d}, \text{ etc.}$$

Thus $$\frac{1}{2} = \frac{2}{4} = \frac{3}{6}, \text{ and } \frac{1 + 2 + 3}{2 + 4 + 6} = \frac{6}{12} = \frac{1}{2} = \frac{2}{4} = \frac{3}{6}.$$

TRANSFORMATIONS OF PROPORTIONS

The properties of fractions and the fundamental principles stated above lead at once to these additional properties of proportions:

362. *If four numbers are in proportion, they are in proportion by* **alternation**; *that is, the first term is to the third term as the second is to the fourth.*

Thus if $a : b = c : d$, then $a : c = b : d$. Why?

363. *If four numbers are in proportion, they are in proportion by* **inversion**; *that is, the second term is to the first as the fourth is to the third.*

Thus if $a : b = c : d$, then $b : a = d : c$. Why?

364. *If four numbers are in proportion, they are in proportion by* **addition**; *that is, the sum of the first two terms is to the second term as the sum of the last two terms is to the fourth term.*

Proof. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} + 1 = \frac{c}{d} + 1$. Why?

Then $$\frac{a+b}{b} = \frac{c+d}{d}.$$

Similarly, $$\frac{a+b}{a} = \frac{c+d}{c}.$$

365. *If four numbers are in proportion, they are in proportion by* **subtraction**; *that is, the difference of the first two terms is to the second term as the difference of the last two terms is to the fourth term.*

Proof. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} - 1 = \frac{c}{d} - 1$. Why?

Then $$\frac{a-b}{b} = \frac{c-d}{d}.$$

Similarly, $$\frac{a-b}{a} = \frac{c-d}{c}.$$

EXERCISES

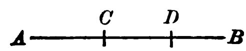
1. In the following proportions determine the value of x:
(a) $9:12 = 8:x$; (b) $4:5 = 12:x$; (c) $7:5 = x:4$; (d) $x:m = c:n$; (e) $\dfrac{a+b}{a-b} = \dfrac{c}{x}$; (f) $12:x = x:27$; (g) $x:a^3b = ab^3:x$.

2. Is the following proportion correct: $3\frac{1}{2} : 4\frac{1}{3} = 5 : 2\frac{1}{4}$?

3. Given the integers 2, 3, 4, 5, 6, 8, 9, 12. Find the fourth proportional to any three chosen at random; the mean proportional between any two; the third proportional to any two.

4. Find the third proportional to m and n; c and d; $a+b$ and $c+d$.

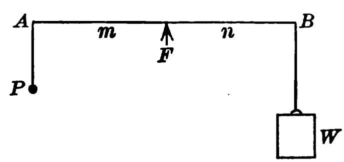
5. Suppose that, in the annexed figure, AB is 30 in. long. Find the lengths of AC, CD, and DB, if $AC:CD = 2:3$, and $CD:DB = 6:5$.



6. Write as a proportion each of the following equations: $cd = mn$; $rs = xy$; $a^2 - b^2 = cd$; $b^2 = ac$; $x^2 = 5yz$; $p^2 = 3q^2$; $x^2 = a^2 - ax$.

7. Two segments are 8 ft. and 12 ft. long respectively. Show that their ratio remains the same if they are measured in inches; in centimeters.

8. The figure represents one form of the straight **lever.** A straight bar AB is supported at F (fulcrum). A weight (W) is attached to it at B. Then, by exerting enough pressure (P) at A, it is possible to keep the bar balanced. Examples of the lever are furnished by the ordinary balance, the crowbar, a pair of scissors. Let the lengths of AF and FB be m and n respectively. Numerous experiments have established the law that if the lever is in equilibrium, $P:W = n:m$; that is, P and W are inversely proportional to m and n. For example, if $AF = 25$ in., and $FB = 20$ in., and a weight of 40 lb. is attached at B, what pressure must be applied at A to raise the weight?



Solution. $P : 40 = 20 : 25.$

$$P = \frac{40 \cdot 20}{25} = 32.$$

$\therefore$ the pressure P is 32 lb.

(No allowance being made for the weight of the lever itself.)

In the following table supply the values of the missing terms:

P	W	m	n
40 lb.	50 lb.	3 ft.	?
35 oz.	70 oz.	?	10 in.
50 kg.	?	40 cm.	20 cm.
?	44 lb.	4 ft.	8 ft.

9. Two weights of 7 lb. and 11 lb. are suspended from the ends of a lever 9 ft. long. Find the position of the fulcrum if the lever is balanced.

10. A weight W is suspended first at A and then at B in the figure for Ex. 8, and the corresponding pressures P_1 and P_2 are found. Show that W is a mean proportional between P_1 and P_2, for any fixed position of F.

11. The sides of a triangle are 7, 8, and 9. Find the segments of 8 made by the bisector of the opposite angle (§ 338).

Suggestion. If m and n are the required segments, then $m : n = 7 : 9$. Hence $m + n : m = 16 : 7$, etc.

12. Show that the side of a square is the mean proportional between the dimensions of an equal rectangle.

13. By transforming each of the following proportions, test the validity of alternation, inversion, addition, subtraction.

$$15 : 10 = 9 : 6,$$
$$a : ab = b : b^2.$$

14. If $a : b = c : d$, show that

$$a + b : a - b = c + d : c - d.$$

In this result a, b, c, and d are said to be in proportion by **addition and subtraction.**

15. Transform by addition and subtraction the proportion

$$a + b : a - b = x + y : x - y.$$

16. If b is the mean proportional between a and c, show that

$$a : c = a^2 : b^2.$$

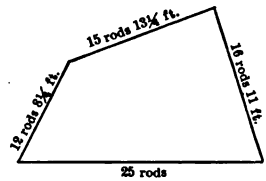
17. If $a : b = c : d$, show that

$$2a^2 + 3b^2 : 2a^2 - 3b^2 = 2c^2 + 3d^2 : 2c^2 - 3d^2.$$

(Let $\frac{a}{b} = \frac{c}{d} = k$; then $a = bk$, and $c = dk$. Substitute these values of a and c in the given proportion and establish an identity.)

SIMILAR POLYGONS

366. A surveyor is sent to measure a tract of land, which is in the form of an irregular polygon, — a quadrilateral, for example. He records the number of feet (or rods) in each side, and the number of degrees in each angle, or at least in as many of the angles as are necessary. He then desires to represent this tract of land by means of a drawing. In order to accomplish this, he uses a scale of a certain number of feet (or rods) to the inch.

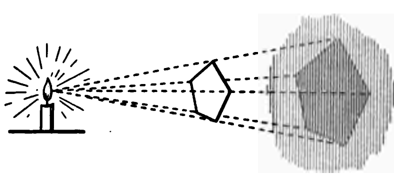


Suppose that the scale is 20 rd. to the inch, and that this figure is the result. Then the ratio of any side in the diagram to the actual length is $1:3960$ $(= 20 \times 16\frac{1}{2} \times 12)$. Has he made any change in the angles?

The drawing conveys a correct idea of the tract of land, for, though smaller in size, it is of the same shape.

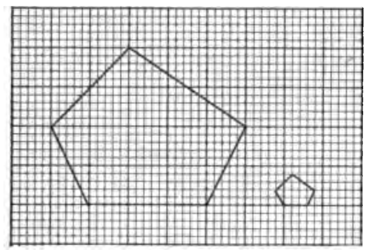
An architect, in drawing the plans for a house, applies the same principle.

367. If a piece of cardboard in the shape of a polygon is held between a source of light and a wall, and parallel to the wall, a shadow is cast which is a copy, on a larger scale, of the original figure.



What relation exists between the angles of the original figure and those of the shadow? What relation between the sides?

368. If a polygon is formed by joining in succession several intersections of the heavy lines of squared paper, and another figure is formed by joining a corresponding series of intersections of the lighter lines, what relation exists between the angles of the two polygons? what relation between the sides?



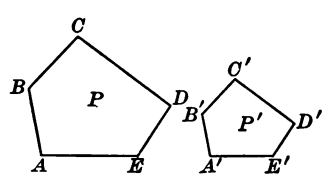
Can the smaller figure be considered a copy of the larger? If so, to what scale?

The *proofs* of the correct answers to the above questions will be derived later.

369. Mutually Equiangular Polygons. Homologous Sides and Angles. Two polygons which have the angles of one equal respectively to the angles of the other are said to be **mutually equiangular.** In that case two of the equal angles (one from each polygon) are known as **homologous angles,** and two sides (one from each polygon) which are similarly situated with reference to the equal angles are called **homologous sides.**

370. Similar Polygons are polygons which have their angles respectively equal and their homologous sides proportional.

The similarity of two figures is indicated by the symbol $\sim$ (a horizontal *s*). Thus $P \sim P'$ means "P is similar to P'."



Similar figures have the same shape.

If P and P' are similar polygons, then

$$\angle A = \angle A', \quad \angle B = \angle B', \quad \angle C = \angle C', \cdots$$

and
$$\frac{AB}{A'B'} = \frac{BC}{B'C'}, \quad \frac{BC}{B'C'} = \frac{CD}{C'D'}, \quad \frac{CD}{C'D'} = \frac{DE}{D'E'}, \cdots$$

371. Ratio of Similitude. From the foregoing considerations it is evident that any two homologous sides of two similar figures may be used to determine the scale by which one figure may be considered a copy of the other. The ratio of two homologous sides of two similar polygons is called the **ratio of similitude** of the polygons.

EXERCISES

1. The sides of a polygon are 8, 9, 10, 12, and 15. Find the sides of a similar polygon if the ratio of similitude of the two polygons is 1 : 2 (2 : 3 ; 3 : 4 ; 4 : 5 ; 5 : 6).

2. The legs of a right triangle are 5 and 12, and the hypotenuse of a similar right triangle is 65. Find the lengths of the legs of the second triangle.

3. Show that two equilateral triangles are similar.

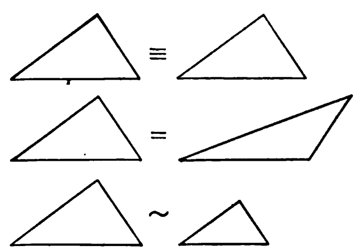
4. Show that two squares are similar; that any two regular polygons of the same number of sides are similar.

5. Construct a rectangle whose sides are 4 cm. and 9 cm. Construct a similar rectangle, the ratio of similitude being 1 : 2.

6. Two rectangular building lots are of the same shape, but each dimension of one is three times the corresponding dimension of the other. Find the ratio of their areas; of their perimeters.

7. If the ratio of similitude of two similar polygons is 1, what additional relation exists between them? In what sense is congruence a special case of similarity?

The adjoining figures illustrate the terms "similarity," "equality," and "congruence," as applied to triangles.

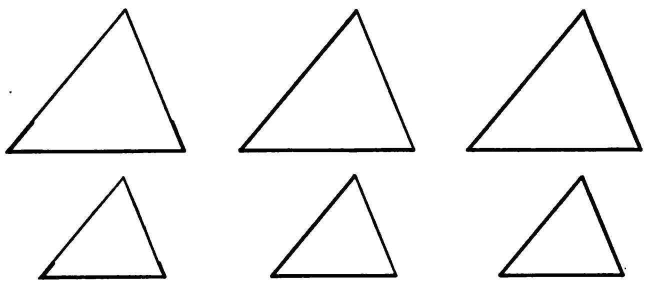


8. Draw a scalene triangle, an irregular quadrilateral, a pentagon, a hexagon. Copy each of them, using in succession the scales 1 : 2, 1 : 3, 2 : 5, 2 : 3. (Is measurement of angles necessary in each case?)

9. In drawing by measurement a polygon similar to a given polygon (see Ex. 8), having given a ratio of similitude, is it necessary to measure (or transfer) all the parts of the given figure? If not, how many parts may be omitted, and what parts?

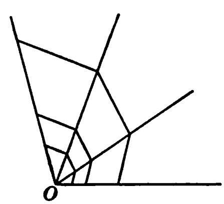
10. How do two similar polygons compare in size, if their ratio of similitude is a proper fraction? unity? an improper fraction?

11. Given the triangle ABC; it is required to construct geometrically another triangle similar to $\triangle ABC$, which shall have to $\triangle ABC$ a ratio of similitude equal to 3 : 4.



Note that this may be done in three different ways, as in the above diagrams, namely, one side and two adjoining angles of the original figure may be used, or two sides and the included angle, or three sides. Several theorems must be established, however, before these constructions can be proved.

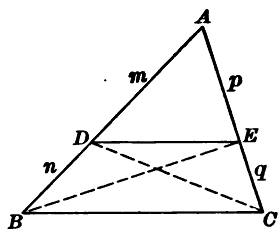
12. A number of straight railroad tracks diverge from the same station at the point O. Several trains leave the station at the same time, one over each track, and run at different rates of speed, say 20, 30, 40, and 50 mi. an hour. Suppose that the trains are stopped simultaneously at the end of 15 min., and again at the end of 30 min., etc. Make a drawing illustrating the statement, using a scale of 5 mi. to the inch. What relation is there between the figures formed by joining in succession the stopping points in each set. Test by measurement with a protractor, or by actual construction, the equality of any of the angles.



PROPORTIONAL SEGMENTS

Proposition I. Theorem

372. *If a line is drawn through two sides of a triangle, parallel to the third side, it divides those sides proportionally.*



Given the triangle ABC, with DE parallel to BC, dividing the sides AB and AC into the segments m and n, and p and q, respectively.

To prove that $m : n = p : q$.

Proof. 1. Draw BE and CD.

2. Then $\triangle ADE : \triangle BDE = m : n$, § 334

and $\triangle ADE : \triangle CDE = p : q$. § 334

3. But $\triangle BDE = \triangle CDE$. Why?

$\therefore \triangle ADE : \triangle BDE = \triangle ADE : \triangle CDE$. Ax. 5

$\therefore m : n = p : q$. Ax. 1

373. The above conclusion may be called the "primary sense" of this theorem. The word "proportionally," however, may be extended in its meaning to include the results obtained by taking the above proportion (§§ 362–364):

by inversion, $n : m = q : p$;

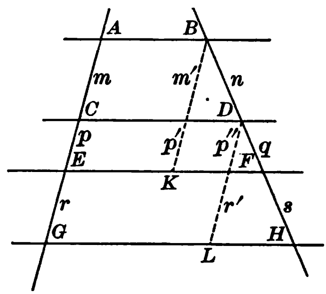
by alternation, $m : p = n : q$;

by addition, $m + n : m = p + q : p$;

that is, $AB : AD = AC : AE$.

This last form of the proportion appears very frequently. In general form it is stated as follows:

374. Corollary 1. *If a line is drawn through two sides of a triangle, parallel to the third side, one of those sides is to either part cut off by the parallel line as the other side is to the corresponding part.*



375. Corollary 2. *If two lines are cut by any number of parallels, the corresponding segments are proportional.*

Draw BK and DL parallel to AG.

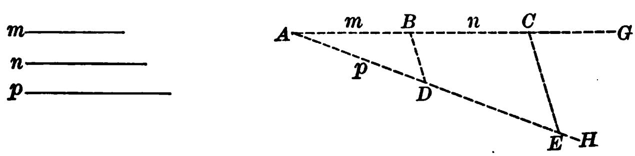
Then $m' = m$, $p'' = p' = p$, and $r' = r$. Why?

But $m' : n = p' : q$, and $p'' : q = r' : s$. § 372

$\therefore m : n = p : q = r : s.$

Proposition II. Problem

376. *To construct the fourth proportional to three given lines.*



Given the lines m, n, and p.

Required *to construct the fourth proportional to m, n, and p.*

Construction. 1. Draw any angle GAH.

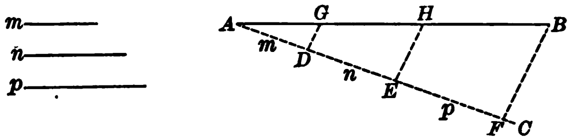
2. On AG take $AB = m$, $BC = n$; on AH take $AD = p$.

3. Draw BD. Through C draw a line parallel to BD, meeting AH in E. Then DE is the required fourth proportional.

Proof. (To be completed.)

Proposition III. Problem

377. *To divide a given line into segments proportional to any number of given lines.*



Given the lines *AB*, *m*, *n*, and *p*.

Required *to divide AB into segments proportional to m, n, and p.*

Construction. 1. Draw AC, making any convenient angle with AB.

(Construction and proof to be completed.)

EXERCISES

1. In how many ways can the construction of § 376 be made? Will the result be the same in each case? (Check by measurement.)

2. Find a third proportional to two given lines m and n.

3. If a, b, c, d, and e are five given lines, how would you construct a line equal to $\frac{bc}{a}$? to $\frac{2\,bc}{a}$? to $\frac{bc}{2\,a}$? to $\frac{bce}{ad}\left[=\frac{b}{a}\left(\frac{ce}{d}\right)\right]$?

4. If a, b, c, and d are four given lines, how would you construct a line equal to $\frac{b^2}{a}$? to $\frac{2\,b^2}{a}$? to $\frac{b^2d}{ac}$?

5. Two lines, a and b, are the dimensions of a rectangle. Construct the altitude of an equal rectangle whose base is equal to a third given line c, without actually constructing the first rectangle.

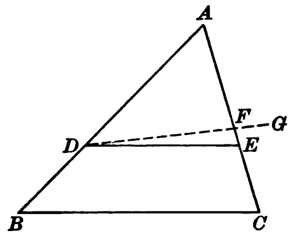
6. Construct the altitude of a rectangle of base b, which is equal to a square of side c, without actually constructing the square.

7. Divide a given square into two rectangles which are to each other as two given lines m and n.

8. If m, n, p, q, and r are the sides of a polygon, and AG is a given line, show how Proposition III may be applied to construct the sides of a polygon similar to the given polygon, with AG as a side homologous to m.

PROPOSITION IV. THEOREM

378. *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*



Given the triangle *ABC*, and the line *DE* drawn so that

$$\frac{AD}{DB} = \frac{AE}{EC}.$$

To prove that $DE \parallel BC$.

Proof. 1. From the given proportion, by addition,

$$AD + DB : AD = AE + EC : AE,$$

or $AB : AD = AC : AE$. § 364

2. Through D draw a line $DG \parallel$ to BC, and suppose that it cuts the line AC in a point F.

3. Then $AB : AD = AC : AF$. § 372

4. But $AB : AD = AC : AE$;

$\therefore AE = AF$, Why?

and the points E and F coincide.

5. $\therefore DE$ coincides with DF. Why?

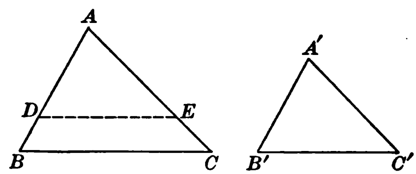
That is, $DE \parallel BC$.

Discussion. Why is the addition form of the proportion used in the above proof? (This theorem is the converse of Proposition I, § 372.)

SIMILAR TRIANGLES

PROPOSITION V. THEOREM

379. *If two triangles have the angles of one equal respectively to the angles of the other, the triangles are similar.*



Given the triangles ABC and $A'B'C'$, having the angles A, B, C equal to the angles A', B', C' respectively.

***To prove** that* $\triangle ABC \sim \triangle A'B'C'$.

Proof. 1. On AB lay off $AD = A'B'$, and on AC lay off $AE = A'C'$. Draw DE.

2. Then $\triangle ADE \equiv \triangle A'B'C'$. Why?

3. Also $DE \parallel BC$. Why?

4. Therefore $AB : AD = AC : AE$; Why?

that is, $AB : A'B' = AC : A'C'$.

5. In like manner, by laying off the corresponding sides of $\triangle A'B'C'$ from B on BA and BC it can be shown that

$$AB : A'B' = BC : B'C'.$$

That is, the homologous sides of $\triangle\triangle ABC$ and $A'B'C'$ are proportional.

6. $\therefore \triangle ABC \sim \triangle A'B'C'$. Why?

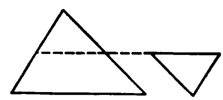
380. COROLLARY 1. *If two triangles have two angles of one equal respectively to two angles of the other, the triangles are similar.*

381. Corollary 2. *If two right triangles have an acute angle of one equal to an acute angle of the other, the right triangles are similar.*

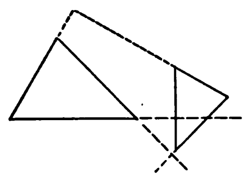
382. Corollary 3. *If two isosceles triangles have the angle at the vertex, or a base angle of one equal to the corresponding angle of the other, they are similar.*

383. Corollary 4. *Two equilateral triangles are similar.*

384. Corollary 5. *If two triangles have their sides respectively parallel, they are similar.*



Produce one or both of two nonparallel sides until they meet. It can then be shown that the triangles have their angles respectively equal.



385. Corollary 6. *If two triangles have their sides respectively perpendicular, they are similar.*

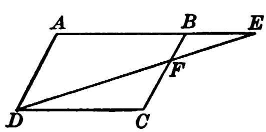
The proof is similar to that of Corollary 5.

Remark. It can be shown that in Corollary 5 and Corollary 6 the parallel sides and the perpendicular sides respectively are homologous sides.

EXERCISES

1. The diagonals of any trapezoid divide each other in the same ratio.

2. A line parallel to one side of a triangle, cutting the other two sides, produced if necessary, forms a triangle similar to the given triangle.

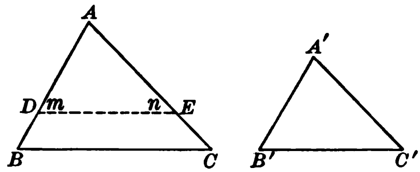


3. In the accompanying figure $ABCD$ is a parallelogram. AB is produced to E. Find all the similar triangles.

4. $ABCD$ is any quadrilateral inscribed in a circle. The diagonals intersect at O. Prove that $\triangle AOB \sim \triangle COD$.

PROPOSITION VI. THEOREM

386. *If two triangles have an angle of one equal to an angle of the other, and the including sides proportional, the triangles are similar.*



Given the triangles ABC and $A'B'C'$ in which the angle A is equal to the angle A' and $AB : A'B' = AC : A'C'$.

To prove that $\triangle ABC \sim \triangle A'B'C'$.

Proof. 1. On AB lay off $AD = A'B'$, and on AC lay off $AE = A'C'$. Draw DE.

2. Then $\triangle ADE \equiv \triangle A'B'C'$. Why?

3. Also $\frac{AB}{AD} = \frac{AC}{AE}$. Why?

4. Therefore $DE \parallel BC$. Why?

Then $\angle m = \angle B$,

and $\angle n = \angle C$. Why?

5. $\therefore \triangle ADE \sim \triangle ABC$. Why?

That is, $\triangle A'B'C' \sim \triangle ABC$. Ax. 1

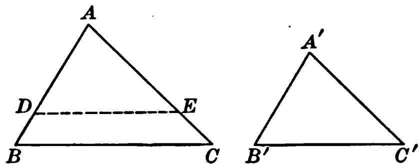
387. COROLLARY 1. *If two right triangles have the legs of one proportional to the legs of the other, they are similar.*

388. COROLLARY 2. *If two right triangles have the hypotenuse and a leg of one proportional to the hypotenuse and a leg of the other, they are similar.*

Since it can be shown by the Pythagorean Theorem that the legs of the two triangles are proportional.

PROPOSITION VII. THEOREM

389. ***If two triangles have their sides respectively proportional, they are similar.***



Given the triangles ABC and $A'B'C'$, in which $AB : A'B' = AC : A'C' = BC : B'C'$.

To prove that $\triangle ABC \sim \triangle A'B'C'$.

Proof. 1. On AB lay off $AD = A'B'$, and on AC lay off $AE = A'C'$. Draw DE.

(It will be proved, first, that $\triangle ADE \sim \triangle ABC$, and then that $\triangle ADE \equiv \triangle A'B'C'$).

2. Then $\triangle ABC \sim \triangle ADE$. § 386

For $\angle A$ is common,

and $AB : AD = AC : AE$. Hyp. and Cons.

3. Hence $\dfrac{AB}{AD} = \dfrac{BC}{DE}$. § 370

4. But $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$; Hyp.

and since $AD = A'B'$, Cons.

$\therefore \dfrac{BC}{DE} = \dfrac{BC}{B'C'}$. Ax. 1

Hence $DE = B'C'$. Why?

5. $\therefore \triangle ADE \equiv \triangle A'B'C'$. s. s. s.

6. $\therefore \triangle ABC \sim \triangle A'B'C'$. Ax. 1

NOTE. Observe that in the above demonstration a congruence is established by means of proportion.

390. Summary. As an aid in working original problems, the results found (§§ 379–389) may be stated as follows:

In order to prove (four lines proportional / two angles equal):

Show that they are homologous (sides / angles) of similar triangles.

In order to prove two triangles similar:

1. Show that their angles are respectively equal.
2. Show that an angle of one is equal to an angle of the other, with the including sides proportional.
3. Show that their sides are respectively proportional.
4. Show that they are right triangles and have
   (a) a pair of acute angles equal, or
   (b) two pairs of corresponding sides proportional.
5. Show that they are isosceles triangles and have
   (a) their vertex angles, or
   (b) their base angles
   } equal.
6. Show that they are both equilateral triangles.
7. Show that their sides are respectively parallel.
8. Show that their sides are respectively perpendicular.

If a, b, c, d, are four line-segments, and it is required to prove that $ad = bc$, show that $a : b = c : d$.

EXERCISES

1. Homologous altitudes of
2. Homologous medians of
3. The radii of circles circumscribed about
4. The radii of circles inscribed in
5. Lines drawn from homologous vertices to the opposite sides and making equal angles with homologous sides in

} similar triangles are proportional to homologous sides.

6. A median of a triangle bisects any line parallel to the corresponding base and included between the other two sides.

7. Lines are drawn parallel to the base of a triangle and are terminated by the other two sides. What is the locus of their middle points?

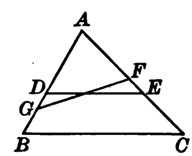
8. Parallel lines are drawn with their extremities in the sides of an angle. Find the locus of their middle points.

9. Given a square $ABCD$. Let E be the middle point of CD, and draw BE. A line is drawn parallel to BE and cutting the square. Let P be the middle point of the segment of this line within the square. Construct the locus of P as the line moves, always remaining parallel to BE.

10. On squared paper plot the points (1, 2), (2, 4), (3, 6). Prove that these points lie on a straight line.

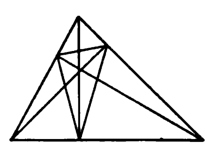
11. If the coördinates of three points are (a, b), (c, d), and (e, f), and $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that the three points lie on a straight line.

12. How many pairs of similar triangles are formed when the three altitudes of any triangle are drawn?

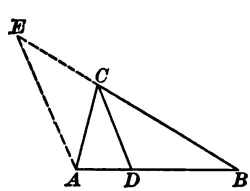


13. Given that $\triangle ADE$ and ABC are similar. Suppose $\triangle ADE$ inverted and $\angle A$ replaced on itself. Then FG, the new position of ED, is said to be **antiparallel** to BC.

Give all the proportions which arise when FG is antiparallel to BC.



14. Show that if the extremities of the three altitudes of a triangle are connected as in the figure, three sets of antiparallel lines are obtained.

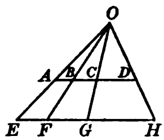


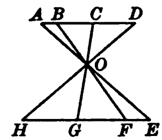
15. Show by the use of antiparallels that the angles of the smaller triangle in Ex. 14 (called the **pedal triangle**) are bisected by the corresponding altitudes.

16. Prove the corollary of Proposition IV, Book III, by drawing AE parallel to CD (see the figure), to meet BC produced in E.

17. State and prove the converse of Ex. 16.

18. A number of rays passing through the same point intercept proportional parts on two parallel lines.





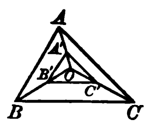
19. In Ex. 18, if $OA : OE = 1 : 2$, what is the value of $OB : OF$? of $OC : OG$?

20. State Ex. 19 as a locus problem.

21. State and prove the converse of Ex. 18.

Suggestion. Let two of the rays, say AE and BF, intersect at O. Then show that a line joining C to O will cut off on the line EH a distance from F equal to FG, and will therefore pass through G and coincide with CG.

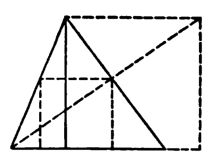
22. The line joining the mid-points of the bases of a trapezoid will, if produced, pass through the point of intersection of the non-parallel sides.



23. ABC is a triangle, and O is any point within or without the triangle. Draw OA, OB, OC. At A', a point in OA, draw $A'B' \parallel$ to AB, and draw $B'C' \parallel$ to BC. Prove that $C'A' \parallel CA$, and that $\triangle A'B'C' \sim \triangle ABC$. O is called the **center of similitude** of the similar triangles ABC and $A'B'C'$.

24. Explain how the method of Ex. 23 might be used to copy a triangle according to a given scale.

25. Given a fixed point D within a triangle ABC. Choose any point E on the perimeter of the triangle, draw DE, and let P be the middle point of DE. Construct the locus of P as E traces out the whole perimeter of the triangle.



26. By studying the annexed figure show how to inscribe a square in a given triangle.

(A square is said to be inscribed in a triangle when one side of the square lies wholly in one side of the triangle, and the other vertices of the square lie in the other two sides of the triangle.)

NUMERICAL EXERCISES

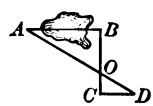
1. Thales of Miletus, a Greek mathematician (600 B.C.), is said to have computed the height of pyramids by means of shadows. He measured the length of the shadows cast at the same time by the pyramid and by an upright rod of known length. Explain.

2. How high is a tree that casts a shadow 60 ft. long, if a vertical post 6 ft. long casts a shadow 8 ft. long at the same time?

3. Complete the following table:

Length of rod	Shadow of rod	Shadow of building	Height of building
3 ft.	5 ft.	60 ft.	
4 ft. 3 in.	2 ft. 10 in.	46 ft.	
6 ft. 8 in.	5 ft.	30 ft.	
10 ft.	11 ft. 4 in.	34 ft.	
9 ft.	2 ft.	9 ft.	

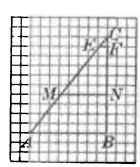
4. Similar triangles may be used to measure distances indirectly.



Suggestion. If AB cannot be measured directly, draw $BC \perp$ to AB, and $CD \perp$ to BC. Place a stake at O, so that AOD is a straight line. Prove that $\triangle OAB \sim \triangle OCD$. What measurements should be taken?

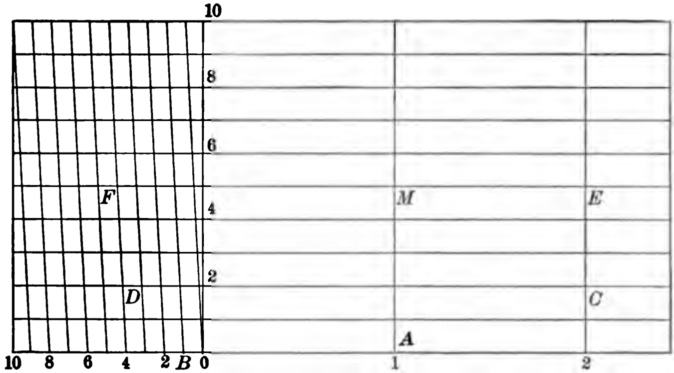
5. The moon is approximately 240,000 mi. from the earth and its diameter is about 2160 mi. How far from the eye must a paper disk 1 in. in diameter be held to cover the moon's disk exactly?

6. The centers of two circles are 10 cm. apart. The radii of the circles are 3 cm. and 2 cm. respectively. How far from the center of the first circle is the line of centers cut by the common interior tangent? by the common exterior tangent? How long are the common tangents?



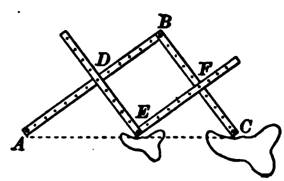
7. The diagram shows how squared paper may be used to divide a small segment into a large number of equal parts, for example, 10. If AB is the given segment, erect $BC \perp$ to AB and equal to 10 divisions. Draw AC. At F, the first division on CB from C, draw $FE \parallel$ to BA. Then $\triangle CEF \sim \triangle ACB$. Prove that EF is one tenth of AB. Similarly, MN is six tenths of AB, etc.

8. The method of division explained in the preceding exercise underlies the construction of the **diagonal scale.** Explain. The scale shown below has one inch for its unit. What is the length of segments AB, CD, EF, MF?



9. Make a diagonal scale and with its aid measure the hypotenuse of a right triangle whose legs are 1 in. and 2 in.; the diagonal of a square of side 1 in.; the altitude of an equilateral triangle of side 2 in. Verify by computation.

10. The **pantograph** is a machine for drawing a plane figure similar to a given plane figure. It was invented in 1603 by Christopher Scheiner. One form of the instrument is shown in the figure. AB and BC, DE and EF, are four bars parallel in pairs and jointed at B, D, E, F. $DEFB$ is a parallelogram.



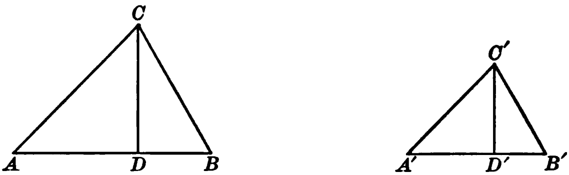
Then, if $\frac{AD}{AB}$ is made equal to $\frac{BF}{BC}$, the points A, E, and C will always be in the same straight line, and the ratio $\frac{AE}{AC}$ will always equal the ratio $\frac{AD}{AB}$.

Now if pencils are placed at E and C while A is a fixed pivot, the points E and C will describe similar figures. Give proof. (Prove that $\triangle ADE \sim \triangle ABC$.)

The pantograph is widely used for reducing and enlarging maps, drawings, etc.

PROPOSITION VIII. THEOREM

391. *The homologous altitudes of two similar triangles have the same ratio as any two homologous sides.*



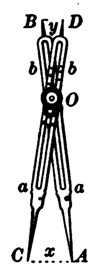
Given the two similar triangles ABC and $A'B'C'$, with the homologous altitudes CD and $C'D'$.

To prove that $\frac{CD}{C'D'} = \frac{AC}{A'C'} = \frac{AB}{A'B'} = \frac{BC}{B'C'}.$

Proof. 1. Rt. triangles CDA and $C'D'A'$ are similar. Why?

(To be completed.)

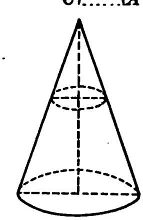
EXERCISES



1. Plot the graph of the equation $\frac{y}{x} = 4$. Prove that it is a straight line.

2. The annexed figure represents a pair of **proportional compasses.** Two equal rods are hinged together at O. Prove that $\triangle AOC \sim \triangle DOB$. Prove that $y : x = b : a$. How may such an instrument be used to construct a triangle similar to a given triangle?

3. The lower and upper bases of a trapezoid are c and d respectively, and the altitude is h. If the legs are extended until they meet, what is the altitude of each of the triangles thus formed?

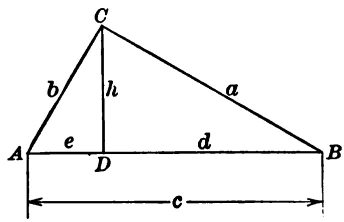


4. The height of a right circular cone* is 10 in. and the diameter of its base is 5 in. What is the diameter of the circular section made by a plane 3 in. from the base?

* A right circular cone is generated by revolving a right triangle about one of its legs as an axis.

PROPOSITION IX. THEOREM

392. *The altitude on the hypotenuse of a right triangle divides the triangle into two triangles each similar to the given triangle, and hence similar to each other.*



Given the right triangle *ABC*, with *CD* perpendicular to the hypotenuse *AB*.

***To prove** that* $rt. \triangle ACD \sim rt. \triangle ABC \sim rt. \triangle BCD$.

Proof. (To be completed.)

If the lines in the above figure are given values represented by the small letters, it follows that

1. $\frac{d}{h} = \frac{h}{e}$. Why?

Hence $h^2 = de$. (1)

2. . $\frac{c}{a} = \frac{a}{d}$. Why?

Hence $a^2 = cd$. (2)

$\frac{c}{b} = \frac{b}{e}$. Why?

Hence $b^2 = ce$. (3)

3. Also $\frac{a^2}{b^2} = \frac{cd}{ce} = \frac{d}{e}$. Dividing (2) by (3).

4. Also $a^2 + b^2 = cd + ce$ Adding (2) and (3).

$= c(d + e)$

$= c^2$.

Statements 1–3 in general form are as follows:

If in a right triangle a perpendicular is let fall from the vertex of the right angle upon the hypotenuse:

393. Corollary 1. *The perpendicular is the mean proportional between the segments of the hypotenuse.*

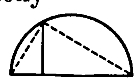
394. Corollary 2. *Either leg is the mean proportional between the whole hypotenuse and the adjacent segment.*

395. Corollary 3. *The squares of the legs are to each other as the adjacent segments of the hypotenuse.*

The Pythagorean Theorem follows also as a corollary (from 4).

396. Corollary 4. *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the legs.*

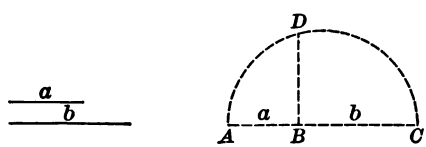
From Proposition IX and Corollary 1 follows directly



397. Corollary 5. *If a perpendicular is let fall from any point in a circle upon any diameter, it is the mean proportional between the segments of the diameter.*

Proposition X. Problem

398. *To construct the mean proportional between two given lines.*



Given the lines *a* and *b*.

Required *to construct the mean proportional between a and b.*

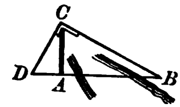
Construction. 1. Draw $AB = a$, and produce AB to C so that $BC = b$.

2. On AC as a diameter describe a semicircle.

3. At B erect a perpendicular upon AC, meeting the circle in D. Then BD is the required mean proportional. Why?

EXERCISES

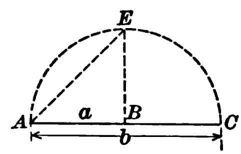
1. A primitive method of determining the distance from a given point A to an inaccessible point B on the same level was based on the use of an instrument that may be improvised by hinging a carpenter's square to the top of a pole set upright in the ground at A. Show how, by measuring AD and AC, to find the distance AB.



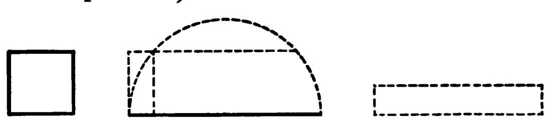
2. If a and b are two given lines, how would you construct $\sqrt{ab}$? $\sqrt{2ab}$? $\sqrt{\frac{ab}{2}}$? $2\sqrt{ab}$? $\frac{1}{2}\sqrt{ab}$? $\sqrt{2a^2}$? $\sqrt{\frac{a^2}{2}}$?

3. Show how Proposition X may be used to transform a rectangle into a square; a polygon into a square. (See §§ 340, 350.)

4. The accompanying figure gives a second construction for the mean proportional (AE) between AB and AC (see § 394). Prove this statement.

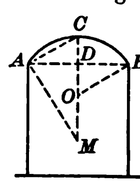


5. To transform a square into a rectangle, the sum of whose base and altitude is equal to a given line. (Describe a semicircle on the given line as a diameter, and at a distance from the diameter equal to one side of the square draw a line parallel to the given line, intersecting the semicircle. From one of the points of intersection let fall a perpendicular on the diameter. The foot of the perpendicular marks off the dimensions of the rectangle. Why? When is the construction impossible?)



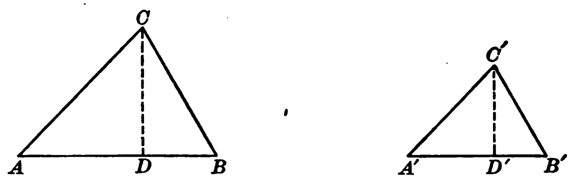
6. To transform a given triangle into an equilateral triangle. (Transform the given triangle into a triangle one of whose angles is 60°, and find a mean proportional between the two sides inclosing the angle.)

7. A door is surmounted by a circular arch. The span AB is 3 ft. (see figure), and the height of the crown CD is 6 in. Find the radius of the circular arch.



Proposition XI. Theorem

399. *The areas of two similar triangles are to each other as the squares of any two homologous sides.*



Given the two similar triangles *ABC* and *A'B'C'*.

To prove that $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Proof. 1. Draw the altitudes CD and $C'D'$.

2. Then $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB \times CD}{A'B' \times C'D'} = \frac{AB}{A'B'} \times \frac{CD}{C'D'}.$$ § 334

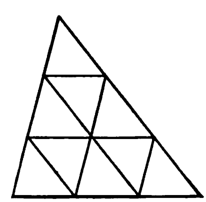
3. But $$\frac{AB}{A'B'} = \frac{CD}{C'D'}.$$ § 391

4. $$\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB}{A'B'} \times \frac{AB}{A'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$ Ax. 1

EXERCISES

1. Prove the above proposition by means of § 337.

2. The accompanying diagram gives a simple illustration of Proposition XI. Explain.



3. If two similar triangles have a ratio of similitude of 2 : 3 [4 : 5; $a : b$], what is the ratio of their areas?

4. If the ratio of the areas of two similar triangles is 4 : 25 [2 : 3; $a : b$], what is their ratio of similitude?

EXERCISES

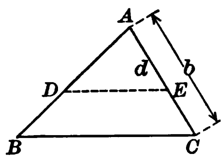
1. The sides of a triangle are 13, 14, 15. Find the three sides of a similar triangle whose area is $2\frac{1}{4}$ times the area of the given triangle.

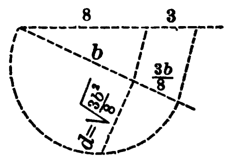
2. On the altitude of a given equilateral triangle as a side a second equilateral triangle is constructed. What is the ratio of the areas of the two triangles?

3. Find the ratio of the areas of two equilateral triangles respectively inscribed in and circumscribed about the same circle.

4. How far from the vertex of a triangle of altitude 10 must a line be drawn parallel to the base to divide the triangle into two equal parts?

5. To divide a triangle in a given ratio by a line parallel to a given side.





Given the $\triangle ABC$.

Required to divide the $\triangle ABC$ in the ratio 3 : 5, by a line ∥ to BC.

Analysis. Suppose the construction completed by means of the line DE.

Then	$\triangle ADE$: trapezoid $DECB = 3 : 5$.	Cons.
Whence	$\triangle ADE : \triangle ABC = 3 : 8$.	By addition
But	$\triangle ADE : \triangle ABC = d^2 : b^2$.	§ 399
	$\therefore d^2 : b^2 = 3 : 8$.	
That is,	$d^2 = \frac{3b^2}{8}$, or $d^2 = b \cdot \frac{3b}{8}$.	

It is therefore necessary first to find a line equal to $\frac{3b}{8}$; that is, to find a fourth proportional to 8, 3, and the side b, and then to find a mean proportional between this line and the side b.

6. Construct DE in Ex. 5, if the given ratio is 4 : 5 (2 : 3, 5 : 7).

7. To divide a triangle into five equal parts by lines parallel to the base.

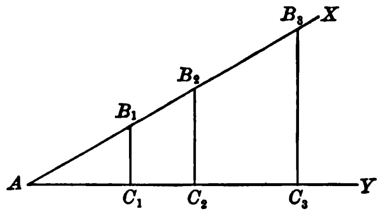
8. At what distances from the vertex should lines be drawn parallel to the base to divide a triangle of altitude m into n equal parts?

TRIGONOMETRIC RATIOS

400. Draw an acute angle XAY. From various points in AX, such as B_1, B_2, B_3, etc., let fall perpendiculars upon the other side, meeting it at C_1, C_2, C_3, etc. A series of similar triangles is formed. (Why? See the figure.)

From this construction it follows that

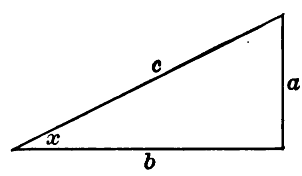
1. $\frac{B_1C_1}{AB_1} = \frac{B_2C_2}{AB_2}$, etc.
2. $\frac{AC_1}{AB_1} = \frac{AC_2}{AB_2}$, etc.
3. $\frac{B_1C_1}{AC_1} = \frac{B_2C_2}{AC_2}$, etc.



The ratios in each group will be equal for all positions of the perpendicular, that is, the ratios will be equal for any number of right triangles containing the angle A.

For example, if $\angle A$ is 30°, the ratio $\frac{BC}{AB}$ for any position of the point B has the value $\frac{1}{2}$.

401. Sine, Cosine, and Tangent of an Angle. Every acute angle, therefore, has corresponding to it a set of definite ratios whose values may be found by constructing any right triangle containing the given angle.



Suppose that $\angle x$ is such an angle, and that a right triangle is constructed to contain it. Let b be the leg of the right triangle adjoining the given angle, a the opposite leg, and c the hypotenuse. Then

The ratio $\frac{a}{c}$ is called the **sine** of x ($\sin x$).

The ratio $\frac{b}{c}$ is called the **cosine** of x ($\cos x$).

The ratio $\frac{a}{b}$ is called the **tangent** of x ($\tan x$).

TRIGONOMETRIC TABLE

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
1°	.017	.9998	.017	45°	.707	.707	1.000
2°	.035	.9994	.035	46°	.719	.695	1.036
3°	.052	.9986	.052	47°	.731	.682	1.072
4°	.070	.9976	.070	48°	.743	.669	1.111
5°	.087	.996	.087	49°	.755	.656	1.150
6°	.105	.995	.105	50°	.766	.643	1.192
7°	.122	.993	.123	51°	.777	.629	1.235
8°	.139	.990	.141	52°	.788	.616	1.280
9°	.156	.988	.158	53°	.799	.602	1.327
10°	.174	.985	.176	54°	.809	.588	1.376
11°	.191	.982	.194	55°	.819	.574	1.428
12°	.208	.978	.213	56°	.829	.559	1.483
13°	.225	.974	.231	57°	.839	.545	1.540
14°	.242	.970	.249	58°	.848	.530	1.600
15°	.259	.966	.268	59°	.857	.515	1.664
16°	.276	.961	.287	60°	.866	.500	1.732
17°	.292	.956	.306	61°	.875	.485	1.804
18°	.309	.951	.325	62°	.883	.469	1.881
19°	.326	.946	.344	63°	.891	.454	1.963
20°	.342	.940	.364	64°	.899	.438	2.050
21°	.358	.934	.384	65°	.906	.423	2.144
22°	.375	.927	.404	66°	.914	.407	2.246
23°	.391	.921	.424	67°	.921	.391	2.356
24°	.407	.914	.445	68°	.927	.375	2.475
25°	.423	.906	.466	69°	.934	.358	2.605
26°	.438	.899	.488	70°	.940	.342	2.747
27°	.454	.891	.510	71°	.946	.326	2.904
28°	.469	.883	.532	72°	.951	.309	3.078
29°	.485	.875	.554	73°	.956	.292	3.271
30°	.500	.866	.577	74°	.961	.276	3.487
31°	.515	.857	.601	75°	.966	.259	3.732
32°	.530	.848	.625	76°	.970	.242	4.011
33°	.545	.839	.649	77°	.974	.225	4.331
34°	.559	.829	.675	78°	.978	.208	4.705
35°	.574	.819	.700	79°	.982	.191	5.145
36°	.588	.809	.727	80°	.985	.174	5.671
37°	.602	.799	.754	81°	.988	.156	6.314
38°	.616	.788	.781	82°	.990	.139	7.115
39°	.629	.777	.810	83°	.993	.122	8.144
40°	.643	.766	.839	84°	.995	.105	9.514
41°	.656	.755	.869	85°	.996	.087	11.43
42°	.669	.743	.900	86°	.9976	.070	14.30
43°	.682	.731	.933	87°	.9986	.052	19.08
44°	.695	.719	.966	88°	.9994	.035	28.64
45°	.707	.707	1.000	89°	.9998	.017	57.29

402. On the opposite page is shown a table which gives the values to three decimal places of the sine, cosine, and tangent of all angles in degrees from 1° to 89°.

To illustrate the use of the Table, find the value of sin 35°. In the first column, under **Angle**, look for 35°, and opposite this in the second column, with the caption **Sin**, we find the number .574. Hence sin 35° = .574.

The Table is also used to find the value of an angle when its sine, cosine, or tangent are given. For example, to find the angle whose tangent equals .933, look for this number in one of the columns with the caption **Tan.** We find .933 opposite 43°. Hence this is the angle sought. If the given number lies between two numbers in the Table, choose the one nearer the given number, and the corresponding angle. For example, for the angle whose cosine equals .432, choose 64°.

What change takes place in the value of the sine as the angle increases? in the value of the cosine? of the tangent?

EXERCISES

1. On a sheet of squared paper and with the aid of a protractor, draw angles of 10°, 20°, 30°, ⋯, 80°, and determine by measurement the approximate values of the sine, cosine, and tangent of these angles, to two decimal places. (Use millimeter paper if possible.) Make a table of these values and compare the results obtained with those given in the Table, p. 260.

2. Find by the Table the values of sin 25°; cos 36°; tan 85°; cos 75°; sin 62°; tan 34°; sin 42°; cos 54°; tan 15°.

3. Find without the use of the protractor or the Table the values of the sine, cosine, and tangent of 30°; of 45°; of 60°. Express the results (in radical form) in a table as follows:

	sine	cosine	tangent
30°			
45°			
60°			

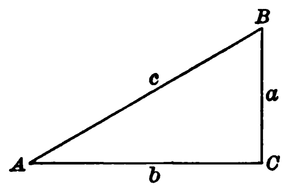
Check the results by reducing to the decimal form and comparing with the Table, given that $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, approximately.

4. Notice that the values of sine and cosine in the Table are less than unity. What is the reason?

APPLICATIONS OF TRIGONOMETRIC RATIOS

403. Since the values of the sine, cosine, and tangent of an angle may be ascertained from the Table, it is possible to find the length of any side of a right triangle, if another side and one of the acute angles are known.

Thus, suppose that in the right triangle ABC the $\angle A$ and the side c are known.

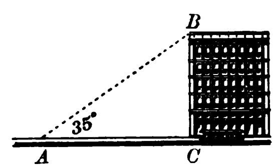


Then, since $\frac{a}{c} = \sin A, \quad \therefore a = c \times \sin A;$

and since $\frac{b}{c} = \cos A, \quad \therefore b = c \times \cos A.$

Similarly, it can be shown that $a = b \times \tan A$, $b = a \times \tan B$, $c = \frac{a}{\sin A}$, etc.

For example, suppose it is desired to find the height of a building without actually measuring it. The horizontal distance AC can readily be measured, and the angle at A can be determined by a transit instrument, or by some simple substitute for it (see § 123). Suppose that AC is 120 ft., and that the angle at A is 35°. Now $BC = AC \cdot \tan A$. From the Table, $\tan A = \tan 35° = .700$. Hence the height of the building, or BC, equals $120 \times .700$, or 84 ft.



If the lengths of two sides of the right triangle are given, and the value of the sine, cosine, or tangent of one of the angles is obtained, the nearest value in the Table under the proper heading gives the value of the angle to the nearest degree.

EXERCISES

In trigonometric problems, when reference is made to the right triangle ABC, it is understood that the $\angle C$ is the right angle, and that the sides opposite the angles A, B, and C are represented by the letters a, b, and c respectively.

By using the Table of values of the trigonometric ratios fill the vacant spaces in the following table:

	a	b	c	A	B
1.	15	. . .	. . .	42°	. . .
2.	. . .	24	. . .	65°	. . .
3.	. . .	. . .	28	. . .	72°
4.	56	. . .	. . .	. . .	50°
5.	. . .	. . .	140	12°	. . .

6. A church steeple subtends an angle of 35° at a point on level ground 90 ft. away. How high is the steeple?

7. A cord is stretched from the top of a pole to a point on the ground 48 ft. from the foot of the pole. It makes an angle of 50° with the horizontal. Find the length of the cord and the height of the pole.

8. A mariner finds that the angle of elevation of the top of a cliff is 16°. He knows from the location of a buoy that his distance from the foot of the cliff is half a mile. How high is the cliff?

9. A ladder resting against the side of a house makes an angle of 24° with the house. If it reaches a point 18 ft. from the ground, how long is the ladder?

10. A man standing 47 ft. from the corner of a house, in line with the north and south wall, finds that the east and west wall subtends an angle of 52°. How long is the east and west wall?

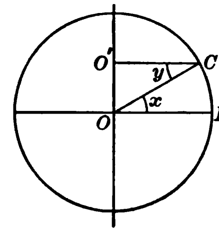
11. The Singer tower, in New York City, is 612 ft. high. How large an angle does it subtend at a point in New York bay, $1\frac{1}{2}$ mi. away? (Assume that the base of the tower is at sea level.)

12. What is the angle of the sun's altitude if a telegraph pole 30 ft. high casts a shadow 42 ft. long?

13. What is the angle of the sun's altitude when a man's shadow is twice his own height?

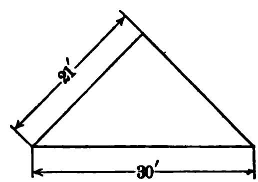
14. Find the radius of a parallel of latitude passing through Chicago (42° N. Lat.), if the radius of the earth is taken as 4000 mi.?

(Note that in the figure $\angle x = \angle y$. Why?)



15. A balloon of diameter 50 ft. is directly above an observer and subtends a visual angle of 4°. What is the height of the balloon?

16. A house 30 ft. wide has a gable roof whose rafters are 21 ft. long. What is the *pitch* of the roof (that is, the angle between a rafter and the horizontal)?



17. What is the area of the gable in the above example? (Find the altitude by trigonometry.)

18. A boat has sailed 20 mi. in a northeasterly direction. What distance east has it made good? What distance north?

19. A hexagonal table top is cut from a circular slab of wood 4 ft. in diameter. Find a side of the hexagon; find its perimeter and its area.

20. An observer standing at the brink of the Grand Cañon of the Colorado finds that the opposite wall of the cañon subtends an angle of 17°. The horizontal distance of the wall from the point of observation is ascertained to be 15,600 ft. How deep is the cañon at that point?

21. Draw an acute triangle ABC, and draw the altitude on AB. Name the sides opposite the angles A, B, and C by the letters a, b, and c respectively, and name the altitude h. Find the value of h in terms of the side a and the angle B, and again in terms of the side b and the angle A. By equating the values thus found prove that

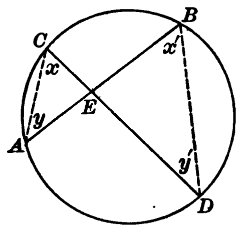
$$a : b = \sin A : \sin B.$$

22. In the acute triangle ABC, $a = 75$ ft., $\angle A = 46°$, and $\angle B = 17°$. Find the side b.

CIRCLES AND PROPORTIONAL LINES

PROPOSITION XII. THEOREM

404. *If two chords intersect within a circle, the product of the segments of one chord is equal to the product of the segments of the other.*



Given in a circle two chords *AB* and *CD* intersecting at *E*.

***To prove** that* $AE \times EB = CE \times ED.$

Proof. 1. Draw AC and BD.

2. Then $\triangle AEC \sim \triangle BED$. § 380

For $\angle x = \angle x'$.

(Each being measured by ½ arc AD.)

And $\angle y = \angle y'$.

(Each being measured by ½ arc CB.)

3. $\therefore AE : ED = CE : EB,$ Why?

or $AE \times EB = CE \times ED.$ § 358

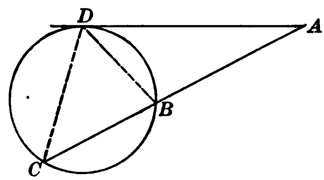
Discussion. Another form of statement of the proposition is: *If two chords intersect within a circle, the segments of one chord are inversely proportional to the segments of the other* (§ 357). One chord through E is a diameter. Draw this diameter and then express the product of the segments of any chord through E in terms of the radius of the circle and the distance from E to the center of the circle.

405. By a **secant from an external point to a circle** is meant the segment of the secant lying between the given external point and the farther point of intersection of the secant and the circle.

The segment between the external point and the nearer point of intersection is called the **external segment** of the secant.

PROPOSITION XIII. THEOREM

406. *If from a point without a circle a secant and a tangent are drawn to the circle, the tangent is the mean proportional between the secant and its external segment.*



Given AD a tangent and AC a secant from the point A to the circle BCD.

***To prove** that* $AC : AD = AD : AB.$

Proof. 1. Draw DC and DB.

2. Then $\triangle ADC \sim \triangle ABD.$ § 380

For $\angle A$ is common;

and $\angle C = \angle ADB.$ Why?

3. $\therefore AC : AD = AD : AB.$ Why?

407. COROLLARY. *If from a fixed point without a circle a secant is drawn, the product of the secant and its external segment is constant, whatever the direction in which the secant is drawn.*

For $AC \times AB = \overline{AD}^2.$

But AD is constant for any fixed position of the point A.

$\therefore AC \times AB$ is constant.

Discussion. In what sense are Propositions XII and XIII and the Corollary to Proposition XIII special forms of one general theorem? State such a theorem.

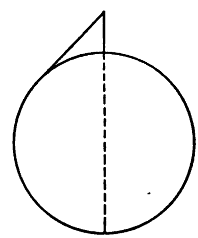
How may Proposition XIII be used to construct a mean proportional between two given lines? Is this method to be preferred to the one already given?

NUMERICAL EXERCISES

1. Draw a circle 2 in. in diameter. Locate within the circle any point not the center. Through this point draw four chords of the circle. Test by measurement the equality of the products of the segments of the chords. (Proposition XII.)

2. A point E is at a distance of 3 in. from the center of a circle whose radius is 5 in. What is the product of the segments of any chord drawn through E?

3. Proposition XIII makes it possible to compute the distance one can see on the earth's surface from a point of known elevation (no allowance being made for refraction or for the condition of the atmosphere). Explain.



4. Assuming that the diameter of the earth is 8000 mi., how far can a man see from the top of a building 100 ft. high? 500 ft. high? how far from the top of a mountain 4000 ft. high?

5. Find the length of a tangent to a circle from a point 15 ft. from the center if the radius of the circle is 9 ft.

6. Find the distance from a point to a circle (§ 268) if a tangent from the point to the circle is 12 ft. long and the radius of the circle is 5 ft.

7. Find the radius of a circle if a tangent from a point 29 ft. from the center is 21 ft. long.

8. Find the radius of a circle if a tangent from a point 8 ft. from the circle is 16 ft. long.

9. Find general formulas for examples like Exs. 5, 7, and 8.

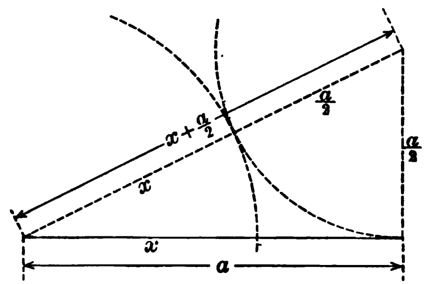
408. Extreme and Mean Ratio. If a line is divided into two segments such that one segment is the mean proportional between the whole line and the other segment, then the line is said to be divided in **extreme and mean ratio.**

That is, the point C divides the line-segment AB in extreme and mean ratio if

$$AB : AC = AC : CB.$$

Proposition XIV. Problem

409. *To divide a given line in extreme and mean ratio.*



Given the segment a.

Required *to construct a segment x such that*

$$a : x = x : a - x.$$

Analysis. Suppose that the construction has been completed. Then

$$x^2 = a(a - x), \qquad \text{Why?}$$

or

$$x^2 = a^2 - ax;$$

that is,

$$x^2 + ax = a^2.$$

Completing the square in the above quadratic equation,

$$x^2 + ax + \left(\frac{a}{2}\right)^2 = a^2 + \left(\frac{a}{2}\right)^2;$$

that is,

$$\left(x + \frac{a}{2}\right)^2 = a^2 + \left(\frac{a}{2}\right)^2.$$

This equation is of the form

$$m^2 = n^2 + p^2.$$

Hence $x + \frac{a}{2}$ is the hypotenuse of a right triangle the legs of which are a and $\frac{a}{2}$, and x is the difference between the hypotenuse and the shorter leg of such a right triangle.

(Construction and proof to be completed.)

410. Corollary 1. *The arithmetical value of x is about .618 of the value of a.*

For, completing the solution of the quadratic equation given above, it is found that $x + \frac{a}{2} = \pm \frac{a}{2}\sqrt{5}$.

Whence, neglecting the negative value of the radical,

$$x = \frac{a}{2}\sqrt{5} - \frac{a}{2}, \text{ or } x = \frac{a}{2}(\sqrt{5} - 1);$$

that is, $x = .618$ of a, nearly.

Note. The segment x in the above construction is called the **major segment**, and the segment $a - x$ the **minor segment** of a.

411. Corollary 2. *If a line which has been divided in extreme and mean ratio is increased at the extremity of its minor segment by a segment equal to its major segment, the resulting line is also divided in extreme and mean ratio.*

For, if $a : x = x : a - x$,

then $a + x : a = a : x$. § 364

Remark. This construction, with the resulting ratio, is celebrated in geometry under the name of the "**Golden Section.**" It is used in the construction of certain regular polygons in Book V. The Golden Section was used by builders during the Middle Ages in many of their architectural designs.

EXERCISES

1. Divide a line 6 in. long in extreme and mean ratio. Find the length of the major segment by measurement and by computation, and compare the results.

2. The mean distances of the earth and the planet Mars from the sun are 92,900,000 mi. and 141,500,000 mi. respectively. Show that when the three bodies are (approximately) in a straight line, the greater distance is divided *nearly* in extreme and mean ratio.

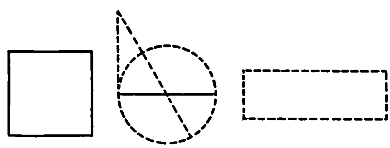
3. The major segment of a line which has been divided in extreme and mean ratio is 3 in. Find the length of the line by construction and measurement (see § 409), and also by computation, and compare the results.

4. The minor segment of a line which has been divided in extreme and mean ratio is 2 in. Find the length of the line by computation, and also by construction and measurement, and compare the results.

5. Measure the length and the width of several books of different sizes, of the surface of a rectangular table, of several rectangular picture frames, of the schoolroom floor. Ascertain whether the Golden Section is apparently used in their construction.

6. In the cathedral of Cologne the height of the towers from the ground to the roof is to their total height as 5 : 8, and the height of the cathedral is to the height of the towers as 21 : 34. What value do these ratios approximate?

7. To transform a given square into a rectangle the difference of whose base and altitude is equal to a given line. (On the given line as a diameter describe a circle. At one end of the diameter erect a perpendicular equal to a side of the given square. From the other end of this perpendicular draw a secant through the center of the circle. What are the dimensions of the rectangle? Is the construction always possible? Why?)



8. To construct a circle which shall pass through two given points and touch a given line.

9. If two circles are tangent externally, the segments of a line drawn through the point of contact and terminating in the circles are proportional to the diameters. Hence show that the corresponding segments of two lines drawn through the point of contact and terminating in the circles are proportional.

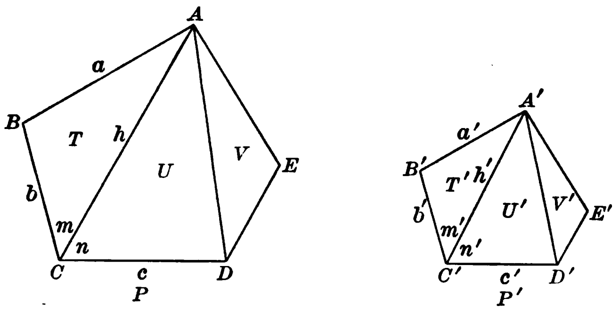
10. If two circles are tangent internally, all chords of the greater circle drawn from the point of contact are divided proportionally by the smaller circle.

11. Two circles touch at M. Through M three lines are drawn, meeting one circle in A, B, C, and the other in D, E, F respectively. Prove that the triangles ABC and DEF, formed by joining the corresponding ends of the lines in succession, are similar.

SIMILAR POLYGONS

Proposition XV. Theorem

412. *If two polygons are similar, they are composed of the same number of triangles, similar each to each and similarly placed.*



Given the similar polygons P and P', with the diagonals from two homologous vertices A and A'.

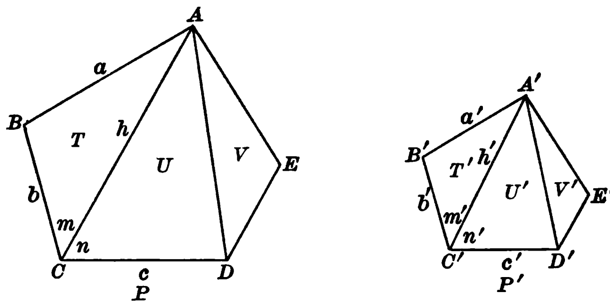
To prove *that* $\triangle T \sim \triangle T'$, $\triangle U \sim \triangle U'$, etc.

Proof. 1.	$\triangle T \sim \triangle T'$.	§ 386
For	$\angle B = \angle B'$,	Why?
and	$a : a' = b : b'$.	Why?
2.	$\therefore \angle m = \angle m'$.	Why?
3. But	$\angle BCD = \angle B'C'D'$.	Why?
	$\therefore \angle n = \angle n'$.	Ax. 3
4. Also	$b : b' = h : h'$,	Why?
and	$b : b' = c : c'$.	Why?
	$\therefore c : c' = h : h'$.	Why?
5.	$\therefore \triangle U \sim \triangle U'$.	Why?

6. In like manner it can be shown that the other triangles are similar, each to each.

PROPOSITION XVI. THEOREM

413. *If two polygons are composed of the same number of triangles, similar each to each and similarly placed, the polygons are similar.*



Given the polygons P and P', with the triangle T similar to the triangle T', the triangle U similar to the triangle U', etc.

To prove that $P \sim P'$.

Proof. 1. $\angle B = \angle B'$. Why?

2. Also $\angle m = \angle m'$, Why?

and $\angle n = \angle n'$.

$\therefore \angle BCD = \angle B'C'D'$. Ax. 2

3. In like manner the other angles are proved to be respectively equal.

4. Also $b : b' = h : h'$, Why?

and $h : h' = c : c'$. Why?

$\therefore b : b' = c : c'$. Ax. 1

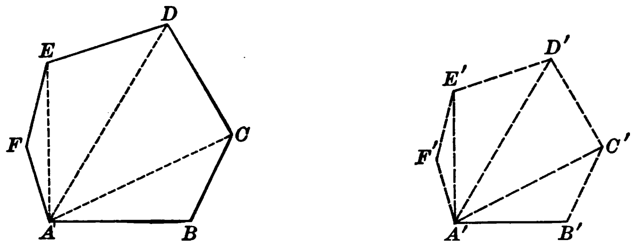
5. In like manner the other homologous sides of the polygons are proved to be proportional.

6. $\therefore P \sim P'$.

(Definition of similar polygons, § 370.)

PROPOSITION XVII. PROBLEM

414. *Upon a given line homologous to one side of a given polygon, to construct a polygon similar to the given polygon.*



Given the line $A'B'$ homologous to AB of the polygon $ABCDEF$.

Required *to construct on $A'B'$ a polygon similar to polygon $ABCDEF$.*

Construction. (Outline.) Draw the diagonals from A. On $A'B'$ homologous to AB construct a triangle similar to $\triangle ABC$. On $A'C'$ homologous to AC construct a triangle similar to $\triangle ACD$. (See § 380.) Repeat this process until for each triangle of the given polygon there is a corresponding similar triangle in the other figure. Then polygon $A'B'C'D'E'F' \sim$ polygon $ABCDEF$. (Why?)

EXERCISES

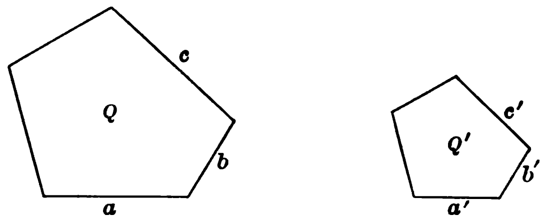
1. Let O be any convenient point within or without the polygon $ABCDE$. Join O to the vertices of the polygon. From A', any point in OA, draw $A'B' \parallel$ to AB, B' lying on OB. Similarly, draw $B'C' \parallel$ to BC, etc. Prove that $E'A' \parallel EA$, and that the new polygon is similar to $ABCDE$.

What name is given to the point O with reference to the two polygons? (See Ex. 23, p. 250.)

2. On coördinate paper mark the points (5, 2), (3, 6), (7, 8), (10, 5), (8, 2). Join these points in succession, forming a polygon. Using the point (0, 0) as a center of similitude (Ex. 23, p. 250), and any desired ratio, construct a polygon similar to the given polygon.

PROPOSITION XVIII. THEOREM

415. *The perimeters of two similar polygons are to each other as any two homologous sides.*



Given the two similar polygons Q and Q', with the perimeters p and p' and the homologous sides a and a'.

To prove that $p : p' = a : a'$.

Proof. 1. $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$, etc. Why?

2. $\therefore \frac{a + b + c + \cdots}{a' + b' + c' + \cdots} = \frac{a}{a'}$. § 361

(In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.)

3. That is, $p : p' = a : a'$.

EXERCISES

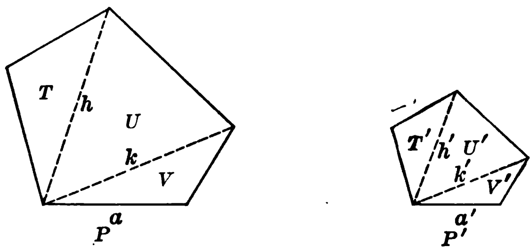
1. Two building lots have the same shape. It takes 400 ft. of fence to inclose one of the lots. If two homologous sides are respectively 40 ft. and 50 ft., how many feet of fence are required to inclose the other lot?

2. A map is drawn to the scale 1:1,000,000. How many miles are represented by the perimeter of a square drawn on the map with a side 1 in. long?

3. The perimeters of two similar triangles are 18 in. and 15 in. One of the altitudes of the first is 6 in. What is the length of the homologous altitude of the other triangle?

Proposition XIX. Theorem

416. *The areas of two similar polygons are to each other as the squares of any two homologous sides.*



Given the two similar polygons P and P', with the areas S and S' and the homologous sides a and a'.

To prove that $S : S' = a^2 : a'^2.$

Proof. 1. Draw the diagonals from two homologous vertices. The two similar polygons are now divided into pairs of similar triangles, T and T', U and U', etc. (Why?)

2. Then $\frac{T}{T'} = \frac{h^2}{h'^2} = \frac{U}{U'} = \frac{k^2}{k'^2} = \frac{V}{V'}.$ § 399

$\therefore \frac{T}{T'} = \frac{U}{U'} = \frac{V}{V'}.$ Ax. 1

3. Hence $\frac{T + U + V}{T' + U' + V'} = \frac{V}{V'}.$ § 361

That is, $\frac{S}{S'} = \frac{V}{V'}.$

4. But $\frac{V}{V'} = \frac{a^2}{a'^2}.$ Why?

$\therefore \frac{S}{S'} = \frac{a^2}{a'^2}.$ Ax. 1

EXERCISES

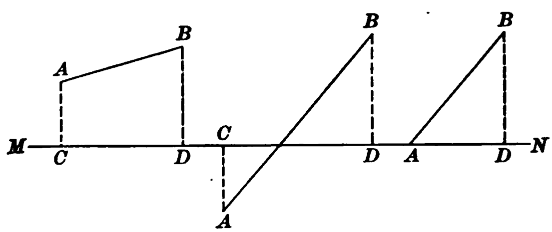
1. The areas of two similar polygons are to each other as 121 : 169. What is the ratio of their perimeters? of their homologous diagonals? of the areas of two corresponding triangles formed by diagonals through homologous vertices? What is the ratio of similitude of two such triangles?

2. To construct a hexagon similar to a given hexagon and having an area twice as great.

3. To construct a polygon similar to two given similar polygons and equal to their sum (or difference). Construct a line (§ 349) whose square is equal to the sum (or the difference) of the squares of homologous sides of the two given polygons. On this line construct (§ 414) a polygon similar to one of the given polygons. Prove the construction.

NUMERICAL PROPERTIES OF TRIANGLES

417. Projection. The **projection of one line upon another** is the segment of the second line included between the feet of the perpendiculars let fall upon it from the extremities of the first line.



CD (in the last figure *AD*) is the projection of *AB* upon *MN*. The line upon which the projection is taken is called the **base line.**

418. *Each leg of a right triangle is the projection of the hypotenuse upon that leg.*

419. *The projection of either leg of an isosceles triangle upon the base is equal to one half the base.*

PROPOSITION XX. THEOREM

420. *In any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides and the projection of the other upon it.*

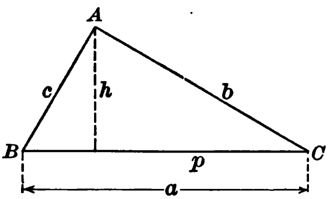


FIG. 1

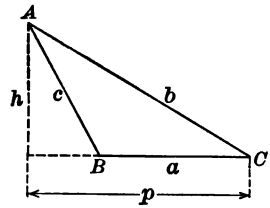


FIG. 2

Given, in the triangle *ABC*, that *p* is the projection of the side *b* upon the side *a*, and that the angle *C* is acute.

***To prove** that* $c^2 = a^2 + b^2 - 2\,ap.$

Proof. 1. Draw the altitude h upon the side a.

2. Then, in Fig. 1, $c^2 = h^2 + (a - p)^2.$ Why?

3. But $h^2 = b^2 - p^2.$ Why?

4. $\therefore c^2 = b^2 - p^2 + a^2 - 2\,ap + p^2$
$= a^2 + b^2 - 2\,ap.$

5. Also, in Fig. 2, $c^2 = h^2 + (p - a)^2.$ Why?

(To be completed.)

EXERCISES

1. What is the projection of AB on BC, if AB is 10 in. long and makes with BC an angle of 30°? 45°? 60°? 90°?

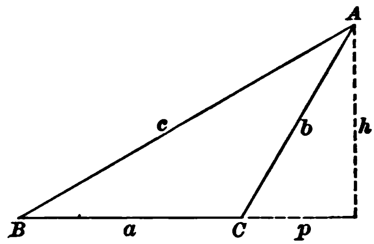
2. Show that the projection of AB upon CD is unchanged if AB is moved toward or away from CD, remaining always parallel to its original position (see fig., p. 276).

3. Show that the projection of AB upon BC is equal to the product of AB and the cosine of the angle B.

4. Show that in the above proposition $c^2 = a^2 + b^2 - 2\,ab \cos C$.

Proposition XXI. Theorem

421. ***In any obtuse triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides increased by twice the product of one of those sides and the projection of the other upon it.***



Given, in the triangle *ABC*, that *p* is the projection of the side *b* upon the side *a*, and that the angle *BCA* is obtuse.

To prove *that* $c^2 = a^2 + b^2 + 2\,ap.$

Proof. 1. Draw the altitude h upon the side a.

2. Then $c^2 = h^2 + (a + p)^2.$ Why?

3. But $h^2 = b^2 - p^2.$ Why?

(To be completed.)

Discussion. When the three sides of a triangle are given, the last two propositions make it possible to find the projection of any side upon any other side. The altitude upon any side may then be determined. Explain (see also § 354).

From Propositions XX and XXI may be derived a method of ascertaining from the lengths of the three sides of a triangle whether the triangle is acute, right, or obtuse.

In a right or an obtuse triangle the greatest side is opposite the right or the obtuse angle. Why? Hence if c is the greatest side of a triangle, and

$c^2 < a^2 + b^2$, the triangle is acute (Proposition XX).

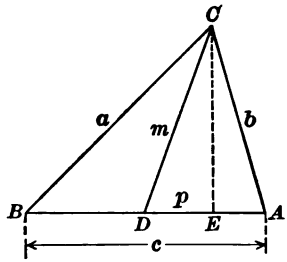
If $c^2 = a^2 + b^2$, the triangle is right.

If $c^2 > a^2 + b^2$, the triangle is obtuse (Proposition XXI).

In the above proposition show that $c^2 = a^2 + b^2 + 2\,ab \cos(180° - C)$.

Proposition XXII. Theorem

422. *In any triangle the sum of the squares of two sides is equal to twice the square of half the third side, increased by twice the square of the median on that side.*



Given the triangle *ABC* in which *m* is the median to the side *c*.

***To prove* that** $$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2\,m^2.$$

Proof. 1. Draw CE perpendicular to AB, and suppose that E falls between A and D. Let DE, the projection of CD upon AB, be denoted by p.

2. Then in $\triangle BCD$, $a^2 = \left(\frac{c}{2}\right)^2 + m^2 + 2\left(\frac{c}{2}\right)p.$ Why?

3. Also in $\triangle ACD$, $b^2 = \left(\frac{c}{2}\right)^2 + m^2 - 2\left(\frac{c}{2}\right)p.$ Why?

4. $\therefore\ a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2\,m^2.$ Ax. 2

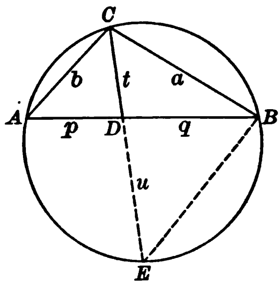
423. Corollary. *In any triangle the difference of the squares of two sides is equal to twice the product of the third side and the projection of the median on that side.*

In the above demonstration $a^2 - b^2 = 2\,cp.$ Ax. 3

424. From the result of § 422 the formula for the length of the median m_c upon any side of a triangle, as c, may be derived, namely,
$$m_c = \tfrac{1}{2}\sqrt{2\,a^2 + 2\,b^2 - c^2}.$$

Proposition XXIII. Theorem

425. *In any triangle the product of two sides is equal to the square of the bisector of the included angle, increased by the product of the segments of the third side made by that bisector.*



Given the triangle *ABC*, the bisector *t* of the angle *C*, and the segments *p* and *q* of the side *c* made by *t*.

To prove that $ab = t^2 + pq.$

Proof. 1. Circumscribe a circle about the $\triangle ABC$. Produce the bisector CD to meet the circumference in E. Draw EB, and let $DE = u$.

2. Then	$\triangle ACD \sim \triangle ECB.$	§ 380
For	$\angle ACD = \angle ECB,$	Hyp.
and	$\angle A = \angle E.$	Why?
3.	$\therefore b : t + u = t : a,$	Why?
or	$ab = t(t + u).$	Why?
That is,	$ab = t^2 + tu.$	
4. But	$tu = pq.$	§ 404
	$\therefore ab = t^2 + pq.$	

426. From the above relation and from § 338 the formula for the length of the bisector t_c upon any side of a triangle, as c, is derived as follows:

Solving for t_c^2,	$t_c^2 = ab - pq.$	
But from § 338,	$p : q = b : a.$	
	$\therefore p : p + q = b : a + b,$	Why?
and	$q : p + q = a : a + b.$	

Since, in the figure, $p + q = c$,

$$\therefore p = \frac{bc}{a+b}, \text{ and } q = \frac{ac}{a+b}.$$

Substituting,
$$t_c^2 = ab - \frac{abc^2}{(a+b)^2}.$$

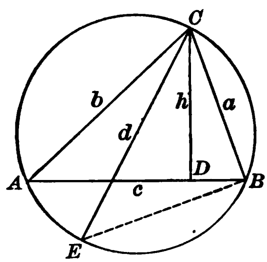
By the use of the method and notation of § 354 it is found that

$$t_c = \frac{2}{a+b}\sqrt{abs(s-c)}.$$

This formula holds for the bisector of the angle C. Analogous formulas hold for the bisectors of angles A and B.

Proposition XXIV. Theorem

427. *In any triangle the product of two sides is equal to the product of the altitude upon the third side by the diameter of the circumscribed circle.*



Given d, the diameter CE of the circle circumscribed about the triangle ABC, and the altitude h upon the side c.

To prove that $ab = hd$.

(To be completed.)

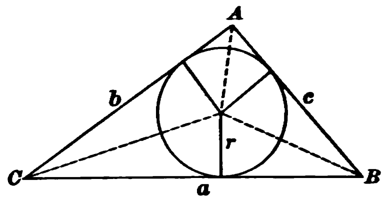
Suggestion. Draw BE and compare △ ACD and ECB.

428. From the above relation, and from the formula (§ 354) for the altitude upon any side of a triangle, the formula for the length of the radius of the circumscribed circle may be derived as follows:

$$R = \frac{d}{2} = \frac{ab}{2h} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

PROPOSITION XXV. THEOREM

429. *In any triangle the area is equal to the product of half the perimeter and the radius of the inscribed circle.*



Given the triangle *ABC*, and the radius *r* of the inscribed circle.

To prove that area $\triangle ABC = \frac{1}{2}(a + b + c)r$.

(To be completed.)

430. From the above relation, and from the formula for the area of a triangle (§ 354), may be derived a formula for the length of the radius of the circle inscribed in any triangle, namely,

$$r = \frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)}.$$

NUMERICAL EXERCISES

1. The sides of a triangle are 13, 14, 15. Find the projection of 15 upon 14, and find the altitude upon 14.

2. In the following table ascertain whether the triangles indicated are acute, right, or obtuse.

a	3	7	5	10	13	21
b	4	9	12	24	15	28
c	6	8	11	26	18	35

3. The sides of a triangle are 6, 7, and 9. Find the projections of 9 and 7 upon 6.

4. Two sides of a triangle are 10 and 12, and they inclose an angle of 45°. Find the projection of 10 on 12, and of 12 on 10. Find the altitude on 12, and the area of the triangle.

5. Two sides of a triangle are 7 and 8, and they inclose an angle of 60°. Find the projection of the side 8 on the side 7, and the third side of the triangle.

6. Two sides of a triangle are 10 and 12, and they inclose an angle of 30°. Find the projection of the side 10 on the side 12, the altitude, and the area of the triangle.

7. Two sides of a triangle are 8 and 9, and the angle between them is 120°. Find the third side and the area.

8. Two sides of a triangle are 18 and 30, the included angle is acute, and the projection of the first upon the second is 12. Find the third side.

9. One side of an acute triangle is 20, and the projection of another side upon it is 10. What is known about this triangle? Is the triangle determined definitely?

10. One side of an acute triangle is 8, and its projection on another side is 4. What is known about this triangle? Is the triangle determined definitely?

11. The sides of a triangle are 9, 12, and 15. Find the three altitudes.

12. The sides of a triangle are 5, 9, and 10. What kind of triangle is it? Find the three altitudes.

13. The sides of a triangle are 14, 16, 18. Find the length of the median on the side 16.

14. The sides of a triangle are 9, 10, and 11. Find the length of the median on the side 9.

15. The sides of a triangle are 14, 48, and 50. Find the area, the altitude on the side 50, and the radius of the circumscribed circle.

16. The legs of a right triangle are 21 and 28. Find the segments of the hypotenuse made by the altitude upon it.

17. Find the altitude on the hypotenuse, and the median on the hypotenuse of the right triangle mentioned in Ex. 16.

18. The sides of a triangle are 8, 26, and 30. Find the radii of the circumscribed and inscribed circles.

19. The sides of a triangle are 6, 7, and 8. Find the length of the bisector of the angle opposite the side 7, terminating in this side.

EXERCISES

Theorems and Locus Problems

1. The difference of the squares of two sides of a triangle is equal to the difference of the squares of the segments made by the altitude upon the third side.

2. The sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals (§§ 204, 422).

3. The sum of the squares of the medians of a triangle is equal to three fourths the sum of the squares of the three sides (§ 422).

4. The sum of the squares of the four sides of any quadrilateral is equal to the sum of the squares of the diagonals increased by four times the square of the line joining the mid-points of the diagonals.

Suggestion. Join the mid-point of one diagonal to the extremities of the other, and apply § 422.

5. If three perpendiculars upon the sides of a triangle meet in a point, the sum of the squares of one set of alternate segments of the sides is equal to the sum of the squares of the other set (§ 344).

6. Find the locus of points whose distances from two fixed parallel lines are in a given ratio.

7. Through a point A on a circle chords are drawn. On each one of these chords a point is taken one third the distance from A to the end of the chord. Find the locus of these points.

8. Given the base of a triangle in magnitude and position and the sum of the squares of the other two sides. Find the locus of the vertex (§ 422).

9. Given the base of a triangle in magnitude and position and the difference of the squares of the other two sides. Find the locus of the vertex (§ 423).

10. Plot the locus of a point if the product of its distances from two perpendicular lines is constant (§ 298).

11. The vertex A of a rectangle $ABCD$ is fixed, and the directions of the sides AB and AD also are fixed. Plot the locus of the vertex C if the area of the rectangle is constant. (See Ex. 10.)

BOOK V

REGULAR POLYGONS AND CIRCLES

Construction of Regular Polygons

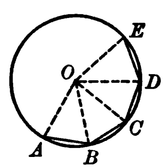
431. Definitions. A polygon that is both equiangular and equilateral is called a **regular** polygon (§§ 57, 95).

A polygon whose sides are chords of a circle is called an **inscribed** polygon (§ 255).

A polygon whose sides are tangent to a circle is called a **circumscribed** polygon (§ 277).

EXERCISES

1. If a series of equal chords are laid off in succession on a circle, what relation exists between the arcs of those chords? between the central angles of those arcs?



2. What relation exists between the angles formed by the successive chords? Give a reason for your answer.

3. Suppose that an inscribed polygon is formed of equal chords of a circle. Why would such a polygon be regular?

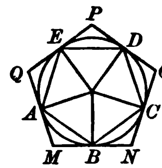
4. How many degrees in the central angle of a regular inscribed polygon of 4 (5, 6, 8, 10, 12, 15, 16) sides? State a general formula for the number of degrees in the central angle of a regular inscribed polygon of n sides.

5. Find the number of degrees in each interior angle of each of the polygons mentioned in the preceding exercise. What relation exists between the central angle and the interior angle of a regular inscribed polygon? Give proof.

6. Which of the angles referred to in Ex. 4 can be constructed geometrically by methods already shown (§§ 156, 158, 188)?

Tabulate the results of the last three exercises.

7. Tangents are drawn to a circle at the vertices of a regular inscribed polygon, forming a circumscribed polygon. A series of triangles is formed by the sides of the two polygons. How are these triangles related to each other? Why are they isosceles?



8. Show that a circumscribed polygon such as is described in the preceding exercise is regular.

Summary

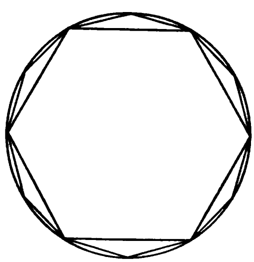
432. *To inscribe in a circle a regular polygon of n sides, construct a central angle of* $\frac{360°}{n}$*, lay off the arc of this angle on the circle n times, and then draw the chords of these arcs.*

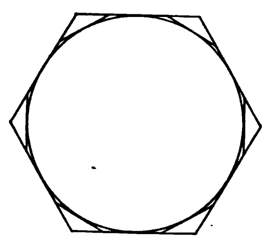
433. *An equilateral polygon inscribed in a circle is regular.*

434. *The central angle of a regular polygon is the supplement of an interior angle at a vertex.*

435. *Tangents drawn at the vertices of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides.*

436. *If the mid-points of the arcs subtended by the sides of a regular inscribed polygon are joined to the adjacent vertices, a regular inscribed polygon of double the number of sides is formed.*



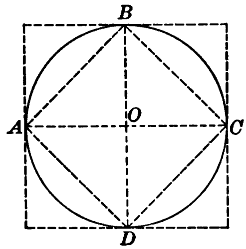


437. *If tangents are drawn at the mid-points of the arcs between adjacent points of contact of the sides of a regular circumscribed polygon, a regular circumscribed polygon of double the number of sides is formed.*

Proposition I. Problem

438. (a) *To inscribe a square in a given circle.*

(b) *To circumscribe a square about a given circle.*



(a) **Given the circle *O*.**

***Required** to inscribe a square in the given circle.*

Construction. 1. Draw two diameters AC and BD perpendicular to each other. § 158

(Construction and proof to be completed.)

439. Corollary. *By bisecting the arcs AB, BC, etc. a regular polygon of 8 sides may be inscribed in the circle; and by continuing the process regular polygons of 16, 32, 64, etc. sides may be inscribed.*

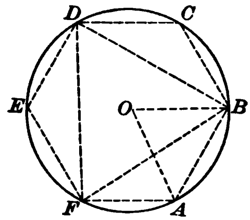
Similarly, regular polygons of 8, 16, 32, 64, etc. sides may be circumscribed about a given circle.

Discussion. Starting with the inscribed square, the construction of § 436 leads by repetition to regular inscribed polygons of 4×2, $4 \times 2 \times 2$, $4 \times 2 \times 2 \times 2$, etc. sides, that is, having 4×2^n sides, n being a positive integer. Since $4 = 2^2$, the Corollary may be stated: *A regular polygon of 2×2^n sides may be inscribed in or circumscribed about a given circle,* where n is any positive integer.

Remark. To inscribe a regular polygon with a given number of sides, it must be possible to construct the central angle $\frac{360°}{n}$ (§ 432) by means of the ruler and compasses. Then the regular polygon can be so constructed and is said to be *geometric;* otherwise not.

PROPOSITION II. PROBLEM

440. *To inscribe a regular hexagon in a given circle.*



(Construction and proof to be supplied.)

Suggestion. $\angle AOB = \frac{360°}{6} = 60°$. $\therefore \triangle AOB$ is equilateral.

441. COROLLARY 1. *By joining the alternate vertices B, D, F, of the regular inscribed hexagon, an equilateral triangle may be inscribed in the circle.*

442. COROLLARY 2. *By bisecting the arcs AB, BC, etc. a regular polygon of 12 sides may be inscribed in the circle; and by repeating the process regular polygons of 24, 48, etc. sides may be inscribed. In other words, regular polygons of* 3×2^n *sides may be inscribed in a given circle, and similarly, regular polygons of* 3×2^n *sides may be circumscribed about a given circle, n being any integer, or zero.*

EXERCISES

1. The radius of a circle is 5 (10, R). How long is the perimeter of a regular inscribed hexagon? of an inscribed square? of a circumscribed square?

2. The perimeter of a regular inscribed hexagon is 42. Find the diameter of the circle.

3. The diameter of a circle is 5 (10, R). Find the area of the inscribed square; of the circumscribed square.

4. The radius of a circle is R. What is the perimeter of the regular circumscribed hexagon?

5. *Analysis of the regular inscribed hexagon:* Prove that

(a) Three of the diagonals are diameters.

(b) The perimeter contains three pairs of parallel sides.

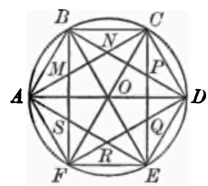
(c) Any diagonal which is a diameter divides the hexagon into two isosceles trapezoids.

(d) Radii drawn to the alternate vertices divide the hexagon into three congruent rhombuses.

(e) The diagonals joining the alternate vertices form an equilateral triangle whose area equals one half the area of the hexagon.

(f) The diagonals joining the corresponding extremities of a pair of parallel sides of the hexagon form with these sides a rectangle.

6. The figure represents a regular hexagon with all its diagonals. Point out the diameters; the parallel lines; the rhombuses; the rectangles; the equilateral triangles; the right triangles; the isosceles triangles; the pairs of lines which bisect each other at right angles; the kites; the equal triangles (for example, $\triangle ABM = \triangle BMN$); a six-pointed star; another regular hexagon.



How many degrees in each angle of the figure?

7. If the radius of a circle is R, find the area of the inscribed equilateral triangle; of the circumscribed equilateral triangle; of the regular inscribed hexagon.

8. The area of the inscribed equilateral triangle equals one fourth the area of the circumscribed equilateral triangle.

9. The area of the regular inscribed hexagon is the mean proportional between the areas of the inscribed and the circumscribed equilateral triangles.

10. In Ex. 6 compare the area of the given hexagon with that of any one of the figures pointed out.

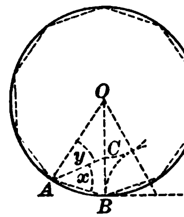
11. If squares are constructed outwardly upon the six sides of a regular hexagon, the exterior vertices of these squares are the vertices of a regular dodecagon.

12. Construct angles of 30°, 15°, $7\frac{1}{2}$°, etc.

13. Construct angles of 45°, $22\frac{1}{2}$°, $37\frac{1}{2}$°, $82\frac{1}{2}$°, $52\frac{1}{2}$°.

Proposition III. Problem

443. *To inscribe a regular decagon in a given circle.*



Given the circle *O*.

***Required** to inscribe a regular decagon in the given circle.*

Analysis. 1. If AB is a side of the required decagon, then $\angle AOB = \frac{360}{10}^\circ = 36^\circ$. Hence $\angle OAB = \angle OBA = 72^\circ$. Why?

2. But $72^\circ = 2 \times 36^\circ$. Draw AC to bisect the $\angle OAB$.

Then in the figure, $\angle x = \angle y = 36^\circ$.

3. This makes the $\triangle ABC$ and AOC isosceles; that is $AB = AC = CO$.

4. Also $\triangle ABC \sim \triangle AOB$. § 382

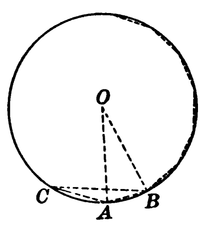
5. $\therefore OB : AB = AB : BC$,

or $OB : OC = OC : BC$.

That is, the radius is divided in extreme and mean ratio (§ 408), and AB ($= OC$) is the major segment of the radius.

(Construction and proof to be completed.)

444. Corollary 1. *The construction of the regular decagon makes possible the construction of regular inscribed or circumscribed polygons of 5, 10, 20, 40 (that is, 5×2^n), sides.*



445. Corollary 2. *To inscribe in a circle a regular pentadecagon, or polygon of fifteen sides.*

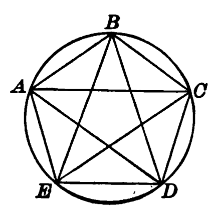
Construct a central angle of 24° by subtracting a central angle of 36° (§ 443) from one of 60° (§ 440). The chord of this angle is the side of a regular inscribed pentadecagon.

446. Historical Note. Propositions I–III establish the fact that a circle can be divided into $2 \times 2^n, 3 \times 2^n, 5 \times 2^n, 15 \times 2^n$, equal parts, n being any positive integer, or zero. The corresponding regular polygons were the only ones known at the time of Euclid, and for centuries it was believed that no other regular polygons could be constructed with ruler and compasses only. In 1796, however, Gauss, a famous German mathematician, found that a circle can be divided into 17 equal parts, using only the instruments named. He also answered the general question, What regular polygons can be constructed geometrically? His result was that the number of sides must be a prime number of the form $2^n + 1$ (as 2, 3, 5, and 17), or a product of different prime numbers of this form (as 15, 51, and 85), or such a prime number or product multiplied by a power of 2 (as 3×2^n or 15×2^n).

EXERCISES

1. The radius of a circle is R. Find the side s_{10} of the regular inscribed decagon. *Ans.* $s_{10} = \frac{R}{2}(\sqrt{5} - 1)$.

2. Draw all the diagonals of a regular inscribed pentagon. How many degrees in each of the angles of the figure? Point out isosceles trapezoids; rhombuses; isosceles triangles.



3. In the figure for Ex. 2, $\angle ACE$ contains 36°. Does that explain how a regular pentagon may be constructed when one of its diagonals is given?

4. The diagonals of a regular pentagon by their points of intersection determine another regular pentagon.

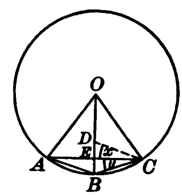
5. In a regular pentagon any two diagonals that are not drawn from the same vertex divide each other in extreme and mean ratio.

6. Construct angles of 36°, 18°, 9°, $4\frac{1}{2}$°.

7. Divide a right angle into five equal parts.

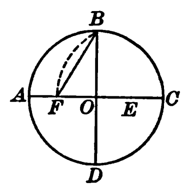
8. In the figure on page 291 prove that $\angle BAE$ is trisected.

9. In the figure let AB $(= BC = OD)$ represent the length of a side of the regular inscribed decagon. Then AC is a side s_5 of the regular inscribed pentagon. Then, since $\angle DCB = 36°$, and $\angle y = 18°$, $\therefore$ $\angle x = 18°$. Hence $DE = EB$. $\therefore$ DE is half the difference between the radius and the side of the decagon. Prove from this relation and from Ex. 1, p. 291, that if the radius is R, $s_5 = \frac{R}{2}\sqrt{10 - 2\sqrt{5}}$.



10. Show that the sum of the squares described upon a side s_{10} of the regular inscribed decagon and a side s_6 of the regular inscribed hexagon equals the square described on a side s_5 of the regular inscribed pentagon (see Exs. 1 and 9).

11. Exercise 10 suggests a short method of constructing a side s_5 of a regular pentagon, as shown in the figure, in which AC and BD are diameters, and $AC \perp BD$. With E, the mid-point of OC, as a center, and with EB as a radius, describe an arc cutting OA at F. Then $OF = s_{10}$ (by Ex. 1, p. 291), $BF = s_5$, and $OB = s_6$. (Ptolemy, 150 A.D.)



12. The top of a table has the form of a regular hexagon (octagon). A carpenter is to strengthen each corner by a triangular strip of wood which fits exactly under the corner. What angles must he give to these pieces of wood if each is in the form of an isosceles triangle?

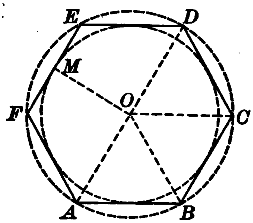
13. Show that an angle of $1\frac{1}{2}°$ can be constructed by means of the ruler and compasses.

14. If it were possible to construct an angle of 1°, the protractor could then be regarded as strictly geometric (see footnote, p. 84). What other regular polygons could then be constructed geometrically?

Historical Note. If any given angle could be trisected, it would be possible to construct a protractor geometrically, for an angle of $1\frac{1}{2}°$ can be constructed. The problem of trisecting an angle is one of the great historical questions which contributed materially to the development of geometry. While a number of angles can readily be trisected, e.g. 90°, 108°, 135°, 180°, a general solution of the problem is impossible by the use of the ruler and compasses alone.

PROPOSITION IV. THEOREM

447. *A circle may be circumscribed about, and a circle may be inscribed in, any regular polygon.*



Given a regular polygon *ABCDEF*.

(a) ***To prove** that a circle may be circumscribed about ABCDEF.*

Proof. 1. It is possible to construct a circle passing through A, B, and C. Let O be its center, and draw OA, OB, OC, and OD.

2. Then	$\triangle AOB \equiv \triangle COD.$	s. a. s.
For	$AB = CD,$	Hyp.
and	$OB = OC;$	Const.
also	$\angle ABC = \angle BCD,$	Why?
and	$\angle OBC = \angle OCB,$	Why?
whence	$\angle ABO = \angle DCO.$	Ax. 3
3.	$\therefore OA = OD.$	Why?

$\therefore$ the circle passing through A, B, C, passes through D.

In like manner it may be proved that this circle passes through E and F.

(b) ***To prove** that a circle may be inscribed in ABCDEF.*

Proof. The sides of the regular polygon are equal chords of the circumscribed circle. Hence they are equally distant from the center. § 264

(To be completed.)

448. The **center** of a regular polygon is the common center of the circumscribed and inscribed circles.

449. The **radius** of a regular polygon is the radius of the circumscribed circle.

450. The **apothem** of a regular polygon is the radius of the inscribed circle.

EXERCISES

If R denotes the radius of a regular inscribed polygon, r the apothem, s one side, A an interior angle, and C the angle at the center, show that

1. In a regular inscribed triangle $s = R\sqrt{3}$, $r = \frac{1}{2}R$, $A = 60°$, $C = 120°$.

2. In an inscribed square $s = R\sqrt{2}$, $r = \frac{1}{2}R\sqrt{2}$, $A = 90°$, $C = 90°$.

3. In a regular inscribed hexagon $s = R$, $r = \frac{1}{2}R\sqrt{3}$, $A = 120°$, $C = 60°$.

4. In a regular inscribed decagon (see Ex. 1, p. 291)

$$s = \frac{1}{2}R(\sqrt{5}-1),\ r = \frac{1}{4}R\sqrt{10+2\sqrt{5}},\ A = 144°,\ C = 36°.$$

5. Show that the perimeter of a regular circumscribed polygon is greater than the perimeter of the regular inscribed polygon of the same number of sides (see § 435).

6. Show that the perimeters and the areas of regular inscribed polygons of 4 (8, 16, 32, etc.) sides form a series of increasing numbers.

7. Show that the perimeters and the areas of regular circumscribed polygons of 4 (8, 16, 32, etc.) sides form a series of decreasing numbers.

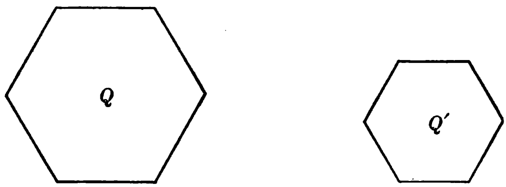
8. Show that the perimeter of a regular circumscribed polygon of 4 (8, 16, 32, etc.) sides is greater than the perimeter of the inscribed square. (Use Exs. 5 and 6 and Ax. 12.)

9. Show that the perimeter of a regular inscribed polygon of 4 (8, 16, 32, etc.) sides is less than the perimeter of the circumscribed square (see Exs. 5 and 7).

10. Prove Exs. 6–9 if the first polygon of the series is one of n sides (n different from 4) and the fixed polygon (Exs. 8 and 9) also has n sides.

PROPOSITION V. THEOREM

451. *Two regular polygons of the same number of sides are similar.*



Given two regular polygons, Q and Q', each having n sides.

To prove that Q *and* Q' *are similar.*

Proof. (To be completed.)

Suggestion. 1. Are the sides of the polygons proportional?
2. Are the polygons mutually equiangular?

452. COROLLARY 1. *The perimeters of two regular polygons of the same number of sides are to each other as any two homologous sides; and the areas of two regular polygons of the same number of sides are to each other as the squares of any two homologous sides.* §§ 415, 416

453. COROLLARY 2. *In a given circle a regular polygon may be inscribed which is similar to any given regular polygon.*

Suggestion. Construct in the given circle a central angle equal to that of the given polygon.

EXERCISES

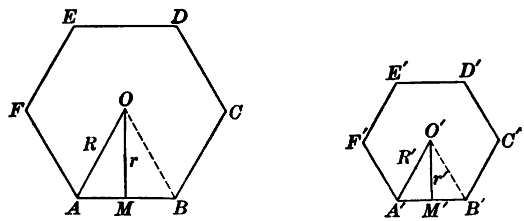
1. On a side of given length to construct a regular polygon of 3 (4, 5, 6, 8, 10, 12, 15) sides.

Suggestion. In an arbitrary circle with center O construct a regular polygon of the required number of sides. Let AB be one side. On the given side $A'B'$ homologous to AB, construct $\triangle A'O'B' \sim \triangle AOB$. Then O' is the center of the new polygon.

2. Through a given point on a given circle draw a chord so as to divide the circle into two arcs having the ratio $1:3$ $(1:5,\ 2:3,\ 3:7,\ 7:8)$.

Proposition VI. Theorem

454. *The perimeters of two regular polygons of the same number of sides are to each other as their radii and also as their apothems.*



Given two regular polygons, each having n sides, and with the perimeters p and p', centers O and O', radii R and R', and apothems r and r'.

To prove that $p : p' = R : R'$,

and $p : p' = r : r'$.

Proof. 1. Draw OB and $O'B'$.

2. Then $\triangle AOB \sim \triangle A'O'B'$. § 382

For $\triangle$s AOB and $A'O'B$ are isosceles, Why?

and $\angle AOB = \angle A'O'B'$.

$\left(\text{Each being equal to } \frac{360°}{n}.\right)$

3. $\therefore AB : A'B' = R : R'$. Why?

Also $AB : A'B' = r : r'$. § 391

4. But $p : p' = AB : A'B'$. § 452

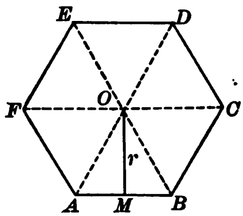
5. $\therefore p : p' = R : R'$, Ax. 1

and also $p : p' :: r : r'$.

455. Corollary. *The areas of two regular polygons of the same number of sides are to each other as the squares of their radii and also as the squares of their apothems.* § 452

PROPOSITION VII. THEOREM

456. ***The area of a regular polygon is equal to half the product of its apothem and its perimeter.***



Given a regular polygon, with the perimeter p, the apothem r, and the area S.

***To prove** that* $S = \frac{1}{2} p \times r.$

Proof. (Outline.)

1. Draw OA, OB, OC, etc., thus dividing the polygon into triangles. These triangles are congruent. Why?

2. Now area of $\triangle AOB = \frac{1}{2} AB \times r$, Why?

area of $\triangle BOC = \frac{1}{2} BC \times r$, etc.

3. $\therefore S = \frac{1}{2}(AB + BC + \cdots) \times r$ Ax. 2

or $S = \frac{1}{2} p \times r.$

EXERCISES

1. If a is a side of a regular hexagon, find its area.

2. If r is the apothem of a regular hexagon, find the area.

3. The area of the regular inscribed octagon is equal to the product of a side of the inscribed square and the diameter of the circumscribed circle.

Solution. Draw radii to three consecutive vertices. Observe that the octagon can be divided into four equal kites, each having for its diagonals one side of the inscribed square and the radius (§ 327).

4. Prove that the area of the regular inscribed dodecagon is equal to three times the square of the radius of the circumscribed circle. (Use the method of Ex. 3.)

CIRCUMFERENCE OF A CIRCLE

PRELIMINARY DISCUSSION

457. The following exercises will serve to introduce the topic to be discussed:

1. Cut out of cardboard a circular disk 4 in. in diameter and wind a thread once around its edge. Measure the length of this thread in inches and divide the number thus obtained by the number of inches in the diameter.

2. Repeat Ex. 1, using disks the diameters of which are 6, 8, and 10 in. Tabulate the results of Exercises 1 and 2 as follows:

Diameter = D	Length of Thread = C	Ratio $C : D$
4		
6		
8		
10		

NOTE. Measure to tenths of an inch and compute to two decimal places.

What do you observe about the ratio $C : D$?

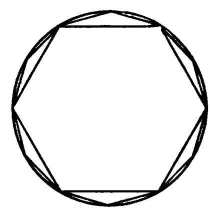
458. Length of the Circle. Hitherto no reference has been made to the "length of a circle" in linear units, for the reason that such a term could have no meaning; it would be impossible to apply a unit straight line to a curved line for purposes of measurement. The expression "length of a circle" calls for a definition all its own. Such a definition will be given in a later section (§ 487). For the present, as is illustrated in the exercises of the preceding section, by the "length of a circle" is meant the length of the straight line which would be obtained if the circle were broken at some point and "straightened out." This form of statement may be regarded as providing a *practical* definition of "circumference."

459. The **circumference** of a circle is its length in linear units.

The attempt to construct, by geometric methods, a straight line which would be equal in length to the circumference of a circle of given radius has occupied the attention of geometers from early times. It is now known that the solution of this problem is not within the means afforded by elementary geometry, and that the numerical ratio of a circumference to its diameter cannot be obtained exactly by algebraic processes.

The method of § 457 is not accurate; nor, if it were, would it be geometric. But an approximation to the length of a circle of given diameter may be made by a method which is accurate and is based upon geometric principles.

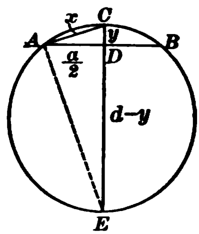
460. Inscribe any convenient regular polygon, say a hexagon, in a circle. Bisect each subtended arc, and join each point of bisection to the two adjoining vertices of the polygon. A regular inscribed polygon of double the original number of sides is obtained. Repeat the process, forming a regular inscribed polygon of quadruple the original number of sides. Suppose now that this process of doubling the number of sides of the regular inscribed polygon were repeated several times. It may be assumed for the present that the lengths of the perimeters of the successive inscribed polygons thus formed approach nearer and nearer to the circumference of the circle, as the number of their sides is increased. By computing these perimeters in succession, it is possible to calculate the circumference of the circle to any desired degree of approximation; that is, to obtain the length of the circle to any required degree of accuracy.



To approximate to the length of a circle, therefore, it is necessary to compute the perimeters of successive regular inscribed polygons each of which has twice as many sides as the preceding polygon. The practical difficulties in making this calculation are greatly lessened by the solution of the following problem:

PROPOSITION VIII. PROBLEM

461. ***Given the diameter of a circle and the side of a regular inscribed polygon, to derive a formula for the side of the regular inscribed polygon of double the number of sides.***



Given CE, the diameter of a circle, AB, a side of a regular inscribed polygon, and CE perpendicular to AB.

Required *to derive a formula for the side AC of a regular inscribed polygon of double the number of sides.*

Solution. 1. Denote AB by a, CE by d, AC by x, and CD by y. Then $AD = \frac{a}{2}$ (why ?), and $DE = d - y$. Draw AE.

2. Then $x^2 = dy$. § 394

3. But $\frac{a^2}{4} = y(d - y) = dy - y^2$. § 393

Transposing, $y^2 - dy + \frac{a^2}{4} = 0$.

The solution of this quadratic equation gives as the value of y,

$$y = \frac{d \pm \sqrt{d^2 - a^2}}{2}.$$

4. But in the case of a regular polygon y is less than $\frac{1}{2}d$. The negative sign before the radical therefore gives the correct value.

Hence, from Step 2, $x = \sqrt{\frac{d^2 - d\sqrt{d^2 - a^2}}{2}}$.

462. Corollary. *If $d = 1$, that is, if a circle of unit diameter is chosen,*

$$x = \sqrt{\frac{1 - \sqrt{1 - a^2}}{2}},$$

or, also,

$$x = \tfrac{1}{2}\sqrt{2 - 2\sqrt{1 - a^2}}.$$

463. The results of computing the perimeters of the polygons mentioned in § 460, by means of the foregoing formula, are shown in the following table:

Regular Polygons inscribed in a Circle whose Diameter is Unity		
Number of Sides	**Length of Side**	**Length of Perimeter**
6	.5	3.
12	.25881904	3.10583
24	.13052619	3.13263
48	.06540313	3.13935
96	.03271908	3.14103
192	.01636173	3.14145
384	.00818114	3.14156
768	.00409060	3.14158

This calculation leads to the conclusion that

The circumference of a circle whose diameter is one linear unit is equal to 3.1416 linear units, nearly.

It has been shown (§ 454) that the perimeters of two regular polygons of the same number of sides are to each other as their radii, and hence also as the diameters of their circumscribed circles. Then it may be assumed that the circumferences C and C' of any two circles have the same ratio as their diameters D and D'. That is, $C : C' = D : D'$. Hence, by alternation (§ 362), $C : D = C' : D'$. It appears, then, that the circumference of *any* circle has the same ratio to its diameter as the circumference of the above circle has to its diameter. In other words, *the ratio of the circumference of a circle to the diameter is constant.* The value of this ratio is represented by the Greek letter π (pronounced "pī"). Hence if C is the circumference of a circle, D its diameter, and R its radius,

$$\frac{C}{D} = \pi,$$

whence $$C = \pi D, \quad \text{and} \quad C = 2\pi R.$$

In the calculation on page 301, $D = 1$. But when $D = 1$, the formula gives $C = \pi$. Hence the last column in the calculation gives successive approximations to the value of π. Using four places of decimals, we take

$$\pi = 3.1416.$$

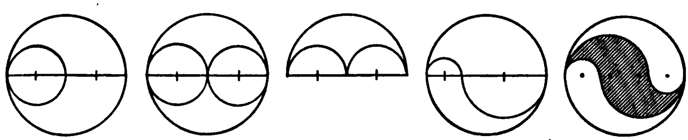
The fraction $\frac{22}{7}$ gives a value of π correct to within $\frac{1}{25}$ of 1%.

464. The length of an arc is found by taking a proportional part of the whole circumference; that is,

The length of an arc of $n°$ is $\frac{n}{360} \times 2\pi R$.

EXERCISES

1. Draw each of the following figures and prove in (1) that the circumference of the small circle is one half that of the large circle; in (2), that the sum of the two small circumferences equals the large circumference. What conclusions may be drawn from (3), (4), (5)?



(1) (2) (3) (4) (5)

2. The length of a circle is 88 (44, 66, a). Find its radius; its diameter. ($\pi = \frac{22}{7}$.)

3. From the following values find the diameter D when the length of its circle C is given, and find C when either R or D is given. C equals 22 (40, 3 in., $4\frac{1}{2}$ ft., 50.5 cm., 2 km.). D equals 3 (50, 2 ft., $1\frac{1}{2}$ in., 9.1 cm., 3 km.). R equals 1 ($1\frac{1}{2}$, 5 ft., 4 ft., $3\frac{1}{2}$ km., 5.1 cm.).

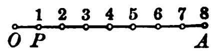
4. One of the famous Sequoia trees in the Mariposa grove in California has a diameter of 32 ft. How large around is it?

5. The perimeter of a circular shaft is 2 ft. What is its diameter?

6. The spoke of a bicycle wheel is 12 in. long. How long is the tire (inside measurement)? If the tire is 1 in. thick, how many times does the wheel revolve in traveling 5 mi.?

7. A circular tower has a circumference of 64 ft. What is its diameter?

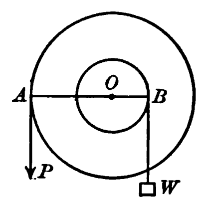
8. The two hands of a clock are 5 in. and 7 in. long respectively. How much greater distance does the extremity of the minute hand travel in one day than that of the hour hand?



9. The radius OA of the flywheel of an engine is 8 ft. During each revolution what fraction of the path described by A is described by a point P, if OP is taken successively equal to 1 ft., 2 ft., etc.?

10. The diameter of a circular race track is 100 yd. How many laps are required to make up a distance of 10 mi.?

11. A weight is to be lifted by means of a wheel and axle. In the figure $OA =$ radius of wheel, $OB =$ radius of axle, $W =$ weight to be raised, $P =$ force applied at A necessary to raise W. It is proved in physics that $\frac{P}{W} = \frac{OB}{OA}$; that is, the force to be applied in order to lift a weight is related to that weight as the radius of the axle is to the radius of the wheel. If $OB = \frac{1}{2} OA$, how is P related to W? Find P, if $W = 100$ lb. and $OA = 2\ OB$.



12. If the weight in Ex. 11 is to be lifted a vertical distance of 50 ft., and the diameter of the axle is 6 in., how many turns of the wheel will lift the weight?

13. A rectangular sheet of paper may be bent so as to form a circular cylinder. One dimension of the rectangle then becomes the height of the cylinder, while the other dimension represents the length of the circle bounding the base of the cylinder. How may the area of such a cylindrical surface be found?

14. A circular tower has a diameter of 20 ft. and a height of 100 ft. Find the area of its cylindrical surface (see Ex. 13).

15. In a certain city there are 150 mi. of street railways. The wires carrying the electric current have a diameter of $\frac{3}{8}$ in. What is the total surface of the wires used for the trolley service?

16. Assuming that in a certain town there are 10,000 telephone poles, find the total surface of these poles, if each has an average diameter of 9 in. and an average height of 35 ft. (ignoring the irregular area of the tops).

17. In Ex. 16, if these poles were to be painted, how long a fence 8 ft. in height could be painted with the same amount of paint?

18. Assuming the earth to be a perfect sphere, how long is the earth's equator if the radius is 4000 mi.?

In Exs. 19–22 C represents the length of a circle and the subscripts signify different circles.

19. Construct a circle equal in length to the sum of two given circles.

Solution. Let x represent the radius of the required circle, while R and R' signify the radii of the given circles.

Then $2\pi x = 2\pi R + 2\pi R'$;

that is, $x = R + R'$.

20. Construct $C = C_1 - C_2$.

21. Construct $C = 3C_1\left(\frac{4C_1}{3}, \frac{5}{8}C_1\right)$.

22. Construct $C = C_1 + C_2 - C_3$.

23. If the radius of a circle is 10, find the side of the inscribed regular polygons of 4, 8, 16, 32 sides.

24. If the side of an inscribed polygon of $2n$ sides is known, the side of an inscribed polygon of n sides can be found by Proposition VIII. Explain.

25. Explain how the side of a regular pentagon may be found if that of the decagon in the same circle is known.

26. Derive formulas for the sides of regular inscribed polygons of 8, 10, 12, 16 sides (§ 461), if the radius is R. Let s_8 be the length of a side of the inscribed polygon of 8 sides, etc.

Ans. $s_8 = R\sqrt{2-\sqrt{2}}$; $s_{10} = \frac{R}{2}(\sqrt{5}-1)$;

$s_{12} = R\sqrt{2-\sqrt{3}}$; $s_{16} = R\sqrt{2-\sqrt{2+\sqrt{2}}}$.

AREA OF A CIRCLE

Preliminary Discussion

465. The following exercises will serve to explain the topic to be considered.

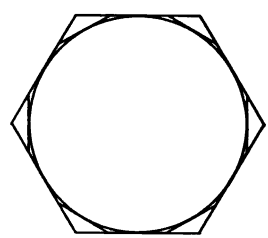
EXERCISES

1. Describe on cross-section paper a circle whose radius is two of the larger units. Count the number of small squares inclosed, reduce the total area found by counting these squares to decimals of a square whose side is the larger unit, and divide by the square of the radius.

2. Repeat the above process, using successive circles with radii of three, four, and five units. Tabulate the results as follows:

Radius = R	Square of Radius	Area = S	Ratio $S : R^2$
2			
3			
4			
5			

The above process is neither accurate nor geometric. A computation which is accurate as well as geometric may be made as follows:



Circumscribe a hexagon about a given circle. Bisect each arc thus formed and draw a tangent at each bisection point. A regular circumscribed polygon of double the number of sides is thus formed. (Why?) Imagine this process repeated several times. The perimeters of the successive polygons thus formed, like those referred to in § 460, may be assumed to approach nearer and nearer the circumference of the circle, and their areas likewise may be assumed to approach nearer and nearer the area of the circle, while the radius of the circle, which is the apothem of each polygon, does not change.

But it is known that the area of each polygon is equal to half the product of its perimeter and its apothem (§ 456). It follows, then, that the area of a circle is equal to half the product of its circumference and its radius.

That is, $S = \frac{1}{2} RC.$

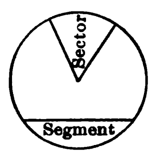
But $C = 2\pi R.$

$$\therefore\ \mathbf{S = \tfrac{1}{2} R \cdot 2\pi R = \pi R^2.}$$

466. *The areas of two circles are to each other as the squares of their radii, or as the squares of their diameters.*

For $\frac{S}{S'} = \frac{\pi R^2}{\pi R'^2} = \frac{R^2}{R'^2} = \frac{D^2}{D'^2}.$

467. A **sector of a circle** is the figure formed by two radii and the arc intercepted between them.

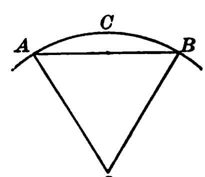


468. *The area of a sector whose central angle contains $n°$ is $\frac{n}{360}\pi R^2$.*

Since the area of the sector bears the same ratio to the area of the circle as its angle bears to the whole angular magnitude about the center, namely, 360°.

469. *The area of a sector is equal to one half the product of its radius and its arc.*

470. A **segment of a circle** is the figure formed by an arc and the chord joining its extremities.



The word "segment" signifies *a part cut off*, and was used in the previous chapters to mean a part of a line. The context will show in each case the sense in which the word is to be taken.

471. The **area of a segment** of a circle (ACB in the figure) can be determined by finding the difference between the area of the sector $O\text{-}ACB$ and the area of the triangle OAB.

EXERCISES

1. If S represents the area of a circle, which has the diameter D and the radius R, find D and R when S is given, and find S when either D or R is given, using the following values:

S equals	4	16	36	5 sq. in.	4.1 sq. cm.	a
D equals	1	2	3	$\frac{1}{2}$ in.	1.2 cm.	$m+n$
R equals	1	2	3	$\frac{1}{3}$ in.	3.7 cm.	$x-y$

2. Determine the effect on the area of a circle, if its radius is multiplied by 2; by 3; by x. Does a similar relation also hold good for diameters?

3. Find the area of a sector if its angle and its radius have the following values. Construct the sector in each case.

Angle	30°	60°	120°	150°	240°	210°	330°	315°	240°
Radius	2 in.	$2\frac{1}{2}$	1.8 cm.	.75 cm.	1.15	2.3	.8 in.	2.6	.33 ft.

4. The lengths of the hands of a watch are in the ratio of 4 : 5. When each has made a revolution, how do the areas of the resulting circles compare?

5. A city has a radius of 3 mi. How many acres in the area of the city?

6. During an earthquake vibrations were noticed 100 mi. from the center of the disturbance. How many square miles were exposed to the effects of the shock?

7. A man learning to ride on a bicycle rides a distance of 1 mi. the first day, $1\frac{1}{2}$ mi. the next day, 2 mi. the third day, etc. How much larger is the territory which is accessible to him on each consecutive day as compared with the previous day, if he extends his trips in this way for a week?

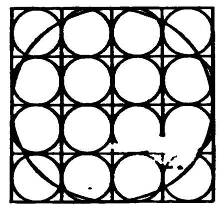
8. A tree has a perimeter of 3 ft. at a certain height. What is the area of its cross section at that height?

9. From a circular piece of tin 8 in. in diameter the largest possible square is to be cut (i.e. the inscribed square). How much waste tin is left?

10. What is the answer in Ex. 9, if a hexagonal piece is to be cut out? an octagonal piece?

11. The volume of a circular cylinder is found by multiplying the area of the circular base by the length of the cylinder. What is the volume of a piece of wire $\frac{1}{4}$ in. in diameter and 20 ft. long?

12. A reservoir constructed for irrigation purposes sends out a stream of water through a pipe 3 ft. in diameter. The pipe is 1000 ft. long. How many times must it be filled if it is to discharge 10,000 acre-feet of water? (An acre-foot of water is the amount of water required to cover 1 A. to a depth of 1 ft.)



13. An ornamental square window is divided into smaller squares, in each of which a circular piece of glass is inserted. Prove that the combined area of the circular pieces is equal to the area of the circle inscribed in the large square.

14. If the diameter of a water pipe is doubled, how many times as much water can the pipe carry in a given time? What seems to be the relation between the increase in the perimeter of the pipe and the increase in its capacity?

15. Construct a circle whose area shall be four times as large as that of a given circle; nine times as large.

In the following exercises S signifies the area of a circle, while the subscripts refer to different circles.

16. Construct $S = S_1 + S_2$.

Solution. Let R_1 and R_2 be the radii of the given circles, and let x be the radius of the required circle.

Then $\pi x^2 = \pi R_1^2 + \pi R_2^2$; that is, $x^2 = R_1^2 + R_2^2$, etc.

17. Construct $S = S_1 - S_2$.

18. Construct $S = 5\,S_1$. [$\pi x^2 = 5\,\pi R^2$. $\therefore x^2 = 5\,R^2$, or $R : x = x : 5\,R$.]

19. Construct $S = S_1 + S_2 + S_3 - S_4$.

20. In a circle whose radius is a, find the area of the smaller segment subtended by each side of a regular inscribed hexagon; of an inscribed square; of an inscribed equilateral triangle.

21. A circular steel shaft of diameter 2 in. is to be ground down so that its diameter becomes $1\frac{1}{2}$ in. What fraction of its volume is to be removed?

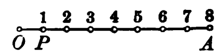
MENSURATION OF THE CIRCLE. FORMAL DEMONSTRATION

VARIABLES AND LIMITS

472. In counting the number of books on a bookshelf one constantly separates the total number of books into two groups, those counted and those not counted. The number of books which have actually been counted is continually increasing, while the number of books that are still to be counted is continually decreasing. The total number of books, however, remains the same. This illustration serves to make clearer the following definitions:

473. A number which has different values during the same discussion is called a **variable.**

474. A number which retains the same value throughout a discussion is called a **constant.**



Let OA represent a rod 8 in. long. Imagine the point P to move from O to A. Then the numbers expressing the lengths of the segments OP and PA are variables, while the total length OA is a constant.

475. Cut a rectangular piece of paper into two equal parts. Lay one of these parts aside and bisect the other. Lay one of these last two pieces aside and bisect the one remaining. Repeat this process a number of times. If the area of the original piece is one square unit, then the areas of the pieces laid aside are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, etc., of a square unit. At any stage in the process the total area of all these pieces will differ from the area of the original piece, namely, one square unit, by the area of the piece about to be bisected. The area of the piece about to be bisected is successively $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, etc. By repeated bisection, *the area of this piece will become and remain less than any assigned positive number, however small.*

Thus it appears that the area of the piece bisected is a variable which approaches 0, while the total area of the other pieces is a variable which approaches 1.

This illustration will serve to introduce the definition of the next section.

476. Limit of a Variable. When the successive values of a variable approach a certain constant number so that the difference between the constant and the variable becomes and remains less than any assigned positive number, however small, then the constant is called the **limit** of the variable.

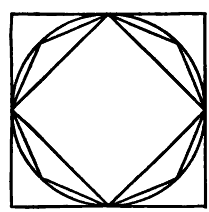
477. The statement "x approaches the limit a," where x is a variable and a is a constant, is sometimes written

$$x \doteq a,$$

the symbol $\doteq$ meaning "approaches the limit."

Application of the Theory of Limits to the Determination of the Length and the Area of a Circle

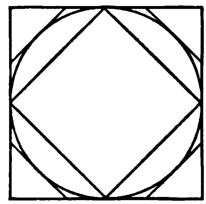
478. About any given circle circumscribe a square. By joining the points of tangency an inscribed square is obtained. (Why?) The perimeter of the inscribed square is less than that of the circumscribed square. (Why?)



Suppose now that the arcs subtended by the sides of the inscribed square are bisected, and that each point of bisection is joined to the adjoining vertices of the inscribed square. A regular inscribed polygon of eight sides is thus constructed, the perimeter of which, as can be shown by the methods of inequalities (Exs. 6 and 9, p. 294), is greater than that of the inscribed square, but less than that of the circumscribed square. If this process of bisecting arcs is repeated several times, a series of regular inscribed polygons is constructed. The perimeters of these polygons increase step by step, but they always remain less than the perimeter of the circumscribed square.

In like manner the areas of the successive polygons increase but always remain less than the area of the circumscribed square.

479. Suppose now that tangents are drawn at the middle points of the arcs subtended by the sides of the inscribed square, to meet the sides of the circumscribed square. Then a circumscribed polygon of eight sides results. Suppose, further, that this process of drawing tangents is continued, the number of sides of the circumscribed polygon being doubled again and again. It can be shown (Exs. 7 and 8, p. 294) that the perimeters and areas of the successive circumscribed polygons decrease but always remain greater than the perimeter and the area respectively of the inscribed square.



480. It is apparent that the above considerations are independent of the number of sides of the regular circumscribed polygon selected at the outset (Ex. 10, p. 294).

481. In the first case the perimeter and the area of the regular inscribed polygon are *increasing* variables, which are always *less* than the constant perimeter and the constant area respectively of a definite regular circumscribed polygon; while in the second case the perimeter and the area of the regular circumscribed polygon are *decreasing* variables, which are always *greater* than the constant perimeter and the constant area respectively of a definite regular inscribed polygon.

Dropping the geometrical language, we have, in the first instance, two variables, each of which constantly *increases* but remains *less* than a given number; and in the second instance, two variables, each of which constantly *decreases* but remains *greater* than a given number. The connection with the Theory of Limits is afforded now by the following assumption:

482. Axiom. Existence of a Limit. ***If a variable constantly increases but always remains less than a given constant, then the variable approaches a limit. This limit is either less than or equal to the given constant.***

Thus if the point P assumes different positions on the line AC so that AP always increases but remains less than AB, then the variable length AP approaches a limiting value AL, which is less than or equal to AB.

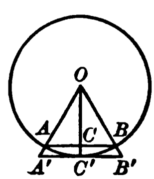
483. It follows that if PC in the above figure constantly decreases but always remains greater than BC, then the variable length PC approaches a limit LC, which is greater than or equal to BC. That is, as a consequence of the above axiom,

If a variable constantly decreases but always remains greater than a given constant, then the variable approaches a limit which is greater than or equal to the given constant.

484. In accordance with the conclusion of § 481 and the axiom of § 482, the perimeter of a regular polygon inscribed in a given circle will approach a definite limit when the number of its sides is doubled an indefinite number of times; and the area of this polygon also will approach a definite limit.

Similarly, by the preceding section, the perimeter of a regular polygon circumscribed about a given circle will approach a definite limit when the number of its sides is doubled an indefinite number of times, and the area also will approach a definite limit.

485. It can now be shown that the perimeter of the inscribed polygon and the perimeter of the similar circumscribed polygon will approach the *same* limit. This is done by proving that the *ratio* of the perimeters of the two polygons approaches *unity* as a limit.



Proof. In the figure let AB and $A'B'$ be homologous sides of two similar regular polygons inscribed in and circumscribed about the same circle. Denote AB by a, and OC' by R, and the perimeters of the two polygons by p and p' respectively.

Then $\frac{p}{p'} = \frac{OC}{OC'} = \frac{\sqrt{\overline{OA}^2 - \overline{AC}^2}}{R} = \frac{\sqrt{R^2 - \frac{1}{4}a^2}}{R}$. Why?

As the number of sides is increased, the side a continually decreases and approaches zero as a limit, the radius R remaining constant. The ratio $\frac{p}{p'}$ increases but remains less than unity. Hence this ratio approaches a limit (§ 482), and the

$$\textit{limit of } \frac{p}{p'} = \frac{\sqrt{R^2 - 0}}{R} = \frac{R}{R} = 1.$$

That is, p and p' have the same limit.

486. In like manner the areas of the two polygons may be shown to approach the same limit.

The foregoing discussion justifies the following definitions:

487. The **circumference of a circle** is the common limit approached by the perimeters of a regular inscribed polygon and the similar regular circumscribed polygon, as the number of sides is continually increased.

488. The **area of a circle** is the common limit approached by the areas of these polygons.

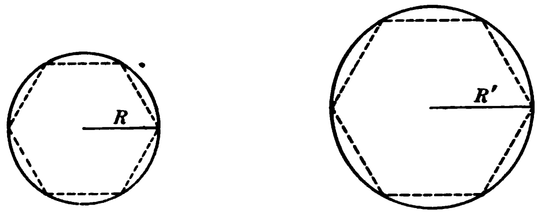
REMARK. There remains, however, the following difficulty. Beginning with the inscribed square, by bisecting arcs and proceeding as in § 478, a series of regular inscribed polygons of 4, 8, 16, 32, etc. sides may be constructed and the perimeters will, by § 482, approach a definite limit. Again, beginning with the regular inscribed hexagon, the same construction leads to a series of regular inscribed polygons of 6, 12, 24, 48, etc. sides, and their perimeters also will approach a definite limit. *Is this limit the same for each of the two series of polygons?* The definition of § 487 *assumes* that these limits are the same. The proof would not be understood at this time, and it is sufficient here to point out that the assumption indicated lurks in the definitions of §§ 487 and 488.

489. The following conclusion is used in the proofs of Propositions IX and X: *If the variable x approaches the limit a, then $\frac{1}{2}x$ will approach the limit $\frac{1}{2}a$, and, in general, $\frac{x}{c}$ will approach the limit $\frac{a}{c}$, c being any constant* (not zero).

The proof follows immediately from the definition of § 476.

PROPOSITION IX. THEOREM

490. *Two circumferences have the same ratio as their radii.*



Given two circles with circumferences C and C', and with radii R and R' respectively.

***To prove** that* $C : C' = R : R'$.

Proof. 1. Inscribe in each circle a regular polygon of n sides, and let p and p' be their perimeters.

2. Then $p : p' = R : R'$, Why?

or $\frac{p}{R} = \frac{p'}{R'}$. Why?

3. Let n increase indefinitely.

Then $\frac{p}{R} \doteq \frac{C}{R}$, §§ 487, 489

and $\frac{p'}{R'} \doteq \frac{C'}{R'}$.

But $\frac{p}{R}$ and $\frac{p'}{R'}$, being always equal variables, are really one and the same variable. Hence their limits are equal.

4. $\therefore \frac{C}{R} = \frac{C'}{R'}$,

or $C : C' = R : R'$.

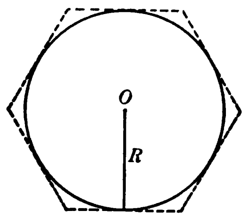
491. COROLLARY 1. *Two circumferences have the same ratio as their diameters.*

492. COROLLARY 2. *The ratio of a circumference to its diameter is constant.*

This constant ratio is denoted by the letter π (§ 463).

Proposition X. Theorem

493. *The area of a circle is equal to half the product of its radius and its circumference.*



Given the circle *O*, of which the area is *S*, the radius *R*, and the circumference *C*.

***To prove** that* $S = \frac{1}{2} C \times R.$

Proof. 1. Circumscribe a regular polygon of n sides about the circle, and let p be its perimeter and A its area. The apothem is equal to the radius of the circle.

2. Then $A = \frac{1}{2} p \times R.$ § 456

3. Let the number of sides of this regular circumscribed polygon increase indefinitely.

Then $A \doteq S,$ § 488

and $\frac{1}{2} p \times R \doteq \frac{1}{2} C \times R.$ §§ 487, 489

But A and $\frac{1}{2} p \times R$, being always equal variables, are really one and the same variable. Hence their limits are equal.

4. $\therefore S = \frac{1}{2} C \times R.$

494. Corollary 1. *The area of a circle is equal to π times the square of the radius.*

495. Corollary 2. *The areas of two circles are to each other as the squares of their radii or of their diameters.*

Discussion. Why is the circumscribed polygon used in the above theorem, while the inscribed polygon is used in Proposition IX?

496. Historical Note. The development of geometry was aided very decidedly by the appearance of problems whose solution seemed to defy the powers of the best mathematicians. Among these difficult problems none has exerted a greater influence, or has attracted greater attention throughout the ages, than the famous problem of "the squaring of the circle." In its original form the problem meant the construction of a square which should have an area equal to that of a given circle. As will now be readily understood, this question involves the determination of π, or the construction of a line equal to the circumference of the given circle. The history of this problem extends over a period of four thousand years, and may be divided into three periods:

1. *From the earliest times to the 17th century A.D.* During this time the value of π was computed by means of polygons, as in Proposition VIII.

2. *From the beginning of modern mathematics (calculus, etc.) to 1766.* During this period the value of π was computed more accurately by *algebraic* methods.

3. *The modern period.* In 1766 Lambert proved that π is irrational. This prepared the way for the modern discovery that π cannot be constructed with ruler and compasses. The proof was given by Lindemann in 1882. *Hence it is impossible to construct geometrically a line equal to the circumference of a given circle, or a square having an area equal to that of a given circle.*

The earliest reference to the number π is found in the Rhind Papyrus (British Museum), written by Ahmes, an Egyptian scribe, about 1700 B.C. His rule for finding the area of a circle consists in squaring eight ninths of the diameter, which makes $\pi = 3.1604$.

The Bible mentions the number π in two places: 1 Kings vii, 23; and 2 Chronicles iv, 2, giving $\pi = 3$ (probably a Babylonian method).

Archimedes (born 287 B.C.), the greatest mathematician of antiquity, by a method similar to that of Proposition VIII, proved that the value of π lies between $3\frac{1}{7}$ and $3\frac{10}{71}$, or, in decimals, between 3.1428 and 3.1408.

Ptolemy, a great astronomer (150 A.D.), found that $\pi = 3\frac{17}{120} = 3.14166$.

The Hindus used $\pi = \sqrt{10} = 3.1623$; also, $\pi = \frac{3927}{1250} = 3.1416$.

Metius of Holland (1625 A.D.) found $\pi = \frac{355}{113} = 3.1415929$, and this decimal is correct through the sixth place.

Ludolph van Ceulen (Leyden, 1610) carried the value of π to thirty-five decimal places.

By improved methods of calculation the value of π has been carried in recent years by Shanks to 707 decimal places. The symbol π, in the present sense, was used for the first time by William Jones, in 1706. It can be shown that π is neither a rational fraction nor a surd. It cannot be expressed exactly by an algebraic number. It is therefore called a **transcendental number.**

The correct value of π to ten places of decimals is 3.1415926535.

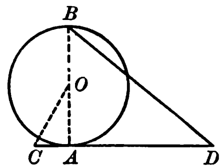
EXERCISES

1. In a certain city whose diameter is about 10 km., it is desired to construct a circular boulevard around the outskirts. Estimate the errors in the length of the inner curb which would arise from using the successive approximations to the value of π in § 463.

2. Plot a graph showing successive approximations of π (§ 463).

Suggestion. On the horizontal axis lay off distances representing the number of sides in the polygons used, and on the vertical axis lay off the corresponding values of π. Use a large scale on the vertical axis.

3. Many attempts have been made to construct a line equal to the length of a circle. The following approximate construction is one of the simplest. It is due to Kochansky (1685). At the extremity A of the diameter AB of a given circle of radius R draw a tangent CD, making $\angle COA = 30°$ and $CD = 3R$. Prove that $BD = R\sqrt{13\frac{1}{3} - 2\sqrt{3}} = 3.1415\,R$; that is, $BD = \frac{1}{2}$ the circumference, very nearly.

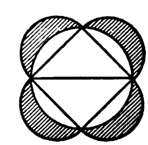


4. Archimedes (250 B.C.) stated the proposition that the area of a circle is equal to that of a triangle whose base is the length of the circle, and whose altitude is the radius. Explain. Construct this triangle approximately by using Ex. 3.

5. Since it is possible to transform any triangle into a square, construct a square approximately equal in area to that of a circle by using the method of Ex. 3.

6. A very old Egyptian manuscript, written about 1700 B.C., contains this rule for finding the area of a circle; that is, for "squaring a circle." From the diameter of a circle subtract one ninth of the diameter, and square the remainder. To what value of π does this construction correspond?

7. Hippocrates (430 B.C.), a great Greek geometrician, tried to "square the circle." The following theorem illustrates his method of approaching the problem: If on the sides of an inscribed square as diameters semicircles are described, the area of the four crescents lying without the circle equals the area of the inscribed square. Give proof.



REVIEW EXERCISES

1. What are the important topics of Book V?

2. What use is made of regular polygons in Book V?

3. Give a brief description of the process by which the length and the area of a circle are determined geometrically.

4. Define π. Prove that it is a constant.

5. Give two formulas for the length of a circle.

6. Give three formulas for the area of a circle.

7. How is the area of a sector found?

8. How is the area of a segment found?

9. If the radius of a circle is multiplied by 2 (3, 4, n), what is the effect on the length of the circle? on the area?

10. The areas of two regular octagons are as $1:16$, and the sum of their perimeters is 25. How long is a side of each?

11. A side of a regular hexagon is 12. Find the radius of the inscribed circle.

12. A circle and a square have equal areas. Which has the greater perimeter?

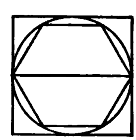
13. A circle and a square have equal perimeters. Which has the greater area?

14. The area of a circle is to be divided into 3 (4, 5, n) equal parts by means of concentric circles. If the radius of the given circle is 100, what are the radii of the concentric circles?

15. The arch of a bridge has the form of a circular arc. Its span (chord) is 280 ft., and the greatest height of the circular arc above its chord is 80 ft. Find the radius of the circle of which the arc is a part.

16. The radius of a circle is 10. Find the area lying between two parallel sides of the inscribed regular hexagon.

17. The apothem of a regular hexagon is 4. Find the area of the hexagon.



18. Find the number of degrees in the central angle of a sector if its perimeter is equal to the circumference of the circle of which it forms a part.

19. Show by the figure that the value of $C:D$ lies between 3 and 4. (Use the perimeters of the polygons.)

MISCELLANEOUS EXERCISES

Composite Figures

TREFOILS

The equilateral triangle may be used as the foundation of numerous ornamental designs, which are called **trefoils**.

These figures are often seen in decorative patterns, and are frequently introduced in the construction of church windows.

1. Construct an equilateral triangle. With each vertex as a center, and with one half of a side as a radius, describe arcs as indicated in Figs. 1 and 2. Let $2a$ represent the length of a side of the equilateral triangle. Find the perimeter and the area of the figure bounded by the arcs.

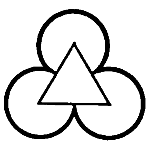


Fig. 1

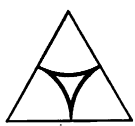


Fig. 2

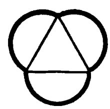


Fig. 3

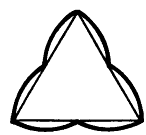
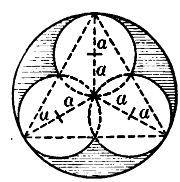


Fig. 4

2. Modify the preceding exercise by using the mid-points of the sides as centers, as indicated in Figs. 3 and 4.

3. Inscribe an equilateral triangle in a circle of radius $2a$. Using the mid-point of each radius of the triangle as a center, and a as a radius, describe circles. A symmetric pattern will result. Find the perimeter and the area of the trefoil and of the shaded part of the figure.



4. The perimeter of a certain church window is made up of three equal semicircles the centers of which form the vertices of an equilateral triangle which has sides $3\frac{1}{2}$ ft. long. Find the area of the window and the length of its perimeter. (Harvard College Entrance Examination paper.)

5. If the area of the trefoil in Fig. 3 above is 50 sq. ft., how long is one side of the equilateral triangle? (Take $\pi = \frac{22}{7}$.)

6. Assuming that the area of the trefoil in each of Figs. 1, 2, and 4 above is 50 sq. ft., find in each case one side of the equilateral triangle.

QUATREFOILS

A square may be used as the foundation of ornamental figures, which are usually called quatrefoils.

7. Construct a square. The length of a side is $2a$. In Figs. 1 and 2 use the vertices as centers and one half of a side as a radius.

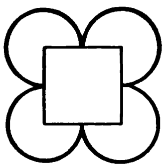


FIG. 1

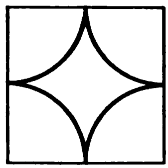


FIG. 2

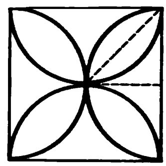


FIG. 3

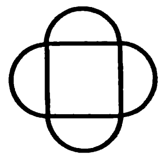
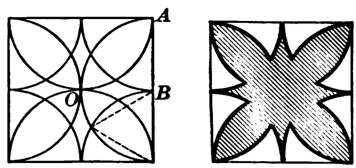


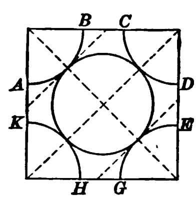
FIG. 4

In Figs. 3 and 4 use the mid-point of each side as a center. In each case find the perimeter and the area of the figure bounded by the arcs. (The construction lines in Fig. 3 show how the area of one lobe of the quatrefoil may be found.)

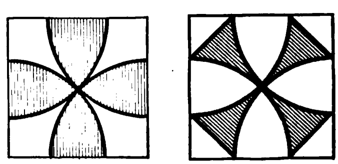
8. The adjoining quatrefoil arises from a combination of Figs. 2 and 3 of the preceding exercise. Find the area and the perimeter of the figure bounded by the arcs. (The construction lines indicate an equilateral triangle, from which the relation of the arcs can be inferred.)



9. In the adjoining figure find the perimeter and the area of $ABCD \cdots K$, if the side of the square is $2a$ and the radius of the circle is $\frac{1}{2}a$.

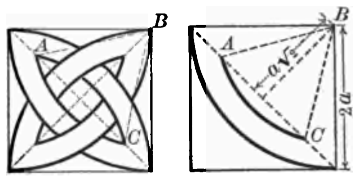


10. The adjoining cross-shaped figures are obtained by using the vertices of the square as centers and one half of a diagonal as a radius. Find the perimeter and the area of each figure if the diagonal equals $2a$.

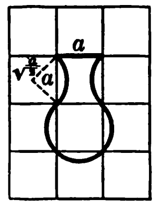


(The second figure was given on page 230 for the purpose of determining the area of a regular octagon.)

11. In the adjoining double quatrefoil the triangle ABC is equilateral. Find the area and the perimeter of the pattern, making no allowance for the overlapping of the two strips. The arcs of each strip are concentric, their centers being the vertices of the square. (Observe that the altitude of $\triangle ABC$ is known. Then the length of AB can be found.)

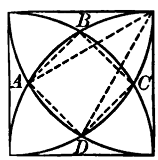


12. On a network of equal squares of side a construct the vaselike figure shown in the diagram. The center of each arc is at the center of a square. Find the perimeter and the area of the figure.



13. The centers of the four arcs in the adjoining figure are the vertices of the square. Prove that $ABCD$ is a square, and find its area, each side of the larger square being $2a$. Find also the area of the quatrefoil whose vertices are A, B, C, and D.

Suggestion. Observe that AD subtends an arc of 30°.

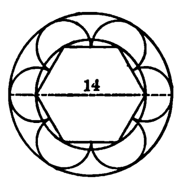


MULTIFOILS

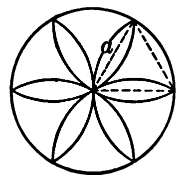
Any regular polygon may be used for the construction of figures similar to those suggested for the square.

14. A rose window of six lobes is to be placed in a circular opening 14 ft. in diameter. It is to have the form shown in the diagram. Determine the radius of the interior hexagon; the perimeter of the hexafoil; the area of the hexafoil.

(How does a radius of one of the smallest circles compare in length with a radius of the largest circle?)



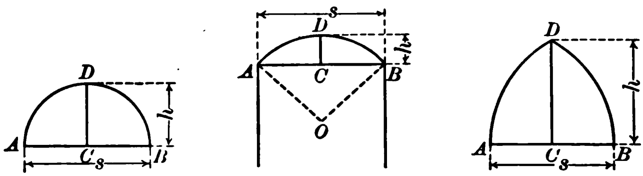
15. About the vertices of a regular inscribed hexagon as centers, and with radii equal to the radius of the given circle, describe arcs within the circle. If the radius of the circle is a, find the perimeter and the area of the resulting hexafoil.



ARCHES

Arches are of great importance in architecture. The most common forms are *semicircular*, *segmental*, and *pointed*.

In the diagrams AB is the width or *span* of the arch, while CD is its *height*. Owing to the symmetry of these arches, numerical



computations involving these forms are greatly simplified. In the following problems, unless the context suggests a different meaning, let

s = span = AB, h = height = CD,

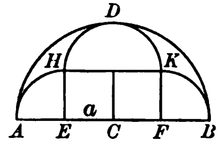
p = perimeter of the arch = length of arc AD + arc BD,

A = area of the arch = area inclosed by p and s.

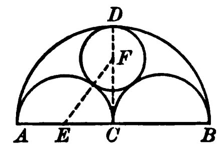
16. Draw a semicircular arch. Let $s = a$. Find h, p, A.

17. The Arch of Triumph in Paris is 162 ft. in total height, 147 ft. in width, and 73 ft. in depth. The crown of its vast arch, which is semicircular, is 96 ft. from the ground. The width of the arch is 48 ft. Find the perimeter of the arch; the area of the interior passage of the arch; the entire exterior surface of the structure, disregarding cornices and other projections.

18. In the adjoining design of a window let $AB = 4a$. Find the length of each arc and the area of each part of the figure.



19. If $AB = 4a$ in the following design, find the radius FD of the upper circle, and then determine the area lying between the circles.



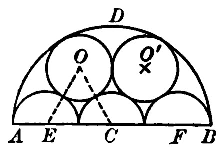
Analysis. Let $FD = x$.

Then $CF = 2a - x$.

But $\overline{EF}^2 = \overline{CE}^2 + \overline{CF}^2$.

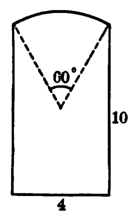
$$\therefore (a + x)^2 = a^2 + (2a - x)^2.$$

$$\therefore x = \frac{2a}{3}.$$



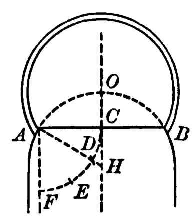
20. In the adjoining window design let $AB = 6a$. O and O' are the vertices of equilateral triangles constructed on CE and CF respectively. Find the area of circles O and O', and the total area of the crescents between the circles.

21. A segmental arch over a door subtends a central angle of 60°. If the door is 10 ft. high and 4 ft. wide, find h, p, and A.



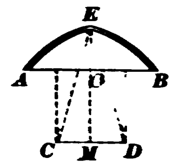
22. The *equilateral Gothic arch* has already been defined (Ex. 13, p. 190). Find the area of an equilateral Gothic arch if $s = 12$.

23. The *horseshoe arch* is used extensively in Moorish architecture. With $AC\left(= \frac{s}{2}\right)$ as a radius and A as a center, construct a quadrant CF. Trisect arc CF at D and E. Draw AD and produce it to meet the perpendicular bisector of AB at H. From H as a center, with radius HA, draw a circle intersecting CH produced at O. Then O is the center of the arch (radius OA). If $s = 2a$, find h, p, and A.



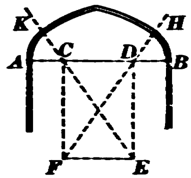
Suggestion. In the rt. $\triangle ACH$, $\angle A = 30°$ and $AC = a$. Find CH and prove $OA = HA$.

24. The *segmental pointed arch* has its centers below the span. In the figure $CM = \frac{1}{2}\,OA$. C and D are the centers of the arcs BE and AE respect[illegible] $s\ a$, and $OM = b$, find ME.

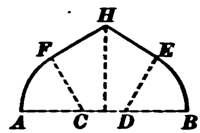


Suggestion. $CE = CB$. Observe that CB is the hypotenuse of a right triangle with legs b and $3\,a$.

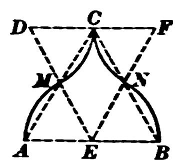
25. A *four-centered arch* may be constructed as follows: Divide the span into four equal parts. On CD construct a rectangle, making $CF = \frac{3}{4}\,AB$. Draw the lines FH and EK. C and D are the centers of the small arcs, and E and F of the larger arcs. If $s = 4\,a$, find EK.



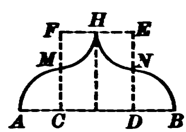
26. The *Brescia arch* (Turkish) is constructed by dividing s into 8 equal parts. On AC and BD, each containing three of these parts, equilateral triangles are constructed. The arcs AF and EB have their centers at C and D respectively. At F and E draw tangents to these arcs meeting in H. If $s = 8\,a$, find h, p, and A. (Draw FE.)



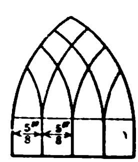
27. The figure represents a *Persian arch.* Triangles ABC and DEF are congruent and equilateral. The centers of the upper arcs MC and NC are respectively D and F, while the lower arcs are drawn with the center E. Prove that the area of the arch equals the area of triangle ABC.



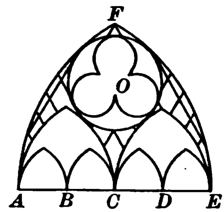
28. In the adjoining modification of the Persian arch the four arcs are equal quadrants. If $s = 4\,a$, prove that the area of the arch is $4\,a^2$.



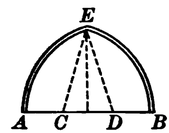
29. Draw the outline of window tracery in the figure, using the given dimensions. The arch is equilateral, and all the arcs are of equal radius. Find the height of each of the pointed arches within the equilateral arch.



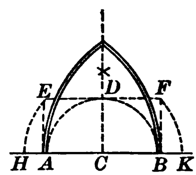
30. In the adjoining diagram, representing a Gothic window, $AB = BC = CD = D$[illegible]. The arches are all equilateral. The point O is [illegible] ection of arcs having the radii AD and E[illegible] (Ex. 14, p. 190). The construction of the trefoil is explained in Ex. 3. If $AB = a$, find the area of each part of the figure, neglecting the tracery.



31. The *drop-pointed arch* is [illegible] by two arcs whose radii are less than the span. In the figure, AB is trisected at C and D. The arcs are constructed with C and D as centers, and radii CB and DA respectively. If $s = 6\,a$, find h. Prove that $h = .645 \times s$ approximately.

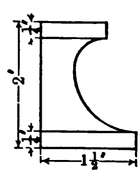


32. The *Early English* or *lancet arch* is formed by two arcs whose radii are greater than the span. In the figure the span AB is bisected at C. Construct the squares CE and CF. Make $CH = CE$, and $CK = CF$. The arcs are constructed with H and K as centers, and with radii HB and KA respectively. If $AB = 2\,a$, find h. Prove that $h = .979 \times s$ approximately.

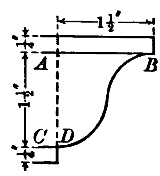


Moldings and Scrolls

The construction of moldings and scrolls is based very largely on the principles of circles that are tangent internally or externally (see Book II, Proposition XIV). These designs are of importance in a number of industries and trades.

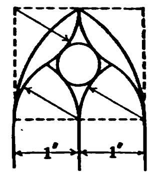


33. Draw the *scotia* molding shown in the margin. The curve is made up of two quadrants of 1 in. and ½ in. radius respectively. How long is the curve?

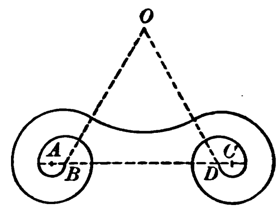


34. Draw the *cyma recta* molding shown in the margin, using the given dimensions. The curve is composed of two quadrants of equal radii, tangent to each other and to the lines AB and CD respectively. How long is the curve?

35. Draw the accompanying diagram of window tracery, using the given dimensions. The arch is equilateral. Explain how the radius of the interior circle is found.

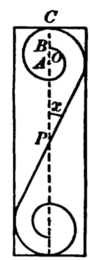


36. Copy the annexed scroll (figure below). It consists of two spirals and a connecting arc. The spirals have two centers, used in succession (Ex. 9, p. 163). The connecting arc is drawn with its center at O, the vertex of an equilateral triangle BOD. If $AB = 5$ mm., and $BD = 7$ cm., find the length of the scroll.



37. Modify the figure in the previous problem by connecting the two spirals by a straight line tangent to each spiral.

38. A balustrade is divided by a series of vertical iron rods into equal rectangular panels. In each of the panels an iron scroll like the one shown in the figure is constructed. The rectangles are 36 in. high and 12 in. wide. Construct one of the panels, using a convenient scale, from the following data. The center of the first semicircle is A, the radius AB being 3 in. O is the mid-point of AB and the center of the second semicircle, the radius OC being $4\frac{1}{2}$ in. From A as a center, with radius AC, draw the next arc, meeting a tangent drawn to it from P, the center of the panel. If $AP = 12$ in., prove that $\angle x = 30°$.



39. If the balustrade in the previous problem contains 20 panels, what is the combined length of the scrolls?

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(References are to page numbers.)

ANNOUNCEMENTS

SHOP PROBLEMS IN MATHEMATICS

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